Amenability and non-uniform growth of some directed automorphism groups of a rooted tree

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Abstract. A result of amenability of some automorphism groups of a spherically homogeneous rooted tree of bounded valency is given. It is used to construct uncountably many amenable groups of non uniform exponential growth. Their Cayley graphs can be made arbitrary close to that of some groups of intermediate growth. Yet those groups are not in the class SG of subexponentially amenable groups.

1. Introduction

Given a finitely generated group Γ endowed with a generating set S the growth function, $b_{\Gamma,S}(r)$ is defined as the number of group elements which are products of less than a given number r of generators and their inverses. The growth of Γ is qualified exponential when the exponential growth rate $h_S(\Gamma) = \lim \sqrt[r]{b_{\Gamma,S}(r)}$ strictly exceeds 1 for some, hence for all, generating set S. The growth is said intermediate if $h_S(\Gamma) = 1$ and the growth function is not polynomial, that is when the group is not virtually nilpotent ([Gro1]). The growth is qualified uniform when the infimum of the exponential growth rates over all generating sets strictly exceeds 1, non uniform when exponential but: $\inf_S h_S(\Gamma) = 1$.

The question of existence of groups of non uniform exponential growth was asked by Gromov in 1981 in the little green book [Gro2]. It has been shown that such groups do not occur in several classes such as hyperbolic groups (see [Kou]), linear groups (see [EMO]), elementary amenable groups (see [Osi2]). A pleasant exposition is given in [dlH]. The first examples of such groups have been provided by Wilson in [Wil1] and [Wil2]. They contain free subgroups. Another example is due to Bartholdi in [Bar]. The main object of this paper is the following:

Theorem 1.1. *There exist uncountably many pairwise non isomorphic amenable groups of non uniform exponential growth.*

These groups will appear as subgroups of the group $Aut(T_{\bar{d}})$ of automorphisms of a spherically homogeneous rooted tree, which is described. In Section 3 a subgroup of $Aut(T_{\bar{d}})$ is proved to be amenable in case of bounded valency of the tree. This Main Theorem 3.1 implies in particular that the group considered in [Bar] is amenable. Sections 4 and 5 are devoted to the proof of this Main Theorem. In Section 6, using specific generating sets of the alternate group of permutation, some groups of intermediate growth are introduced. These groups are proved to be dense in

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the profinite group of alternate automorphism of the rooted tree. The groups of Theorem 1.1 are constructed in Section 7, using results of Wilson ([Wil2]). Some part of Wilson Theorem 7.1, namely the convergence to 1 of the exponential growth rates associated to different generating sets, is reinterpreted as a convergence of the Cayley graphs to Cayley graphs of the groups of intermediate growth introduced in the previous section. The last Section 8 deals with the question of subexponential amenability. The groups of non uniform exponential growth constructed are proved not to be in the class SG.

2. Automorphisms of spherically homogeneous rooted trees

2.1. **Spherically homogeneous rooted tree.** Given a sequence $\bar{d} = \{d_j\}_{j\geq 0}$ of integers $d_j \geq 2$, the associated spherically homogeneous rooted tree denoted $T_{\bar{d}}$ is defined as follows: the vertices are indexed by all finite sequences $v = (i_1 i_2 \dots i_k)$ with i_j in $\{1, 2, \dots, d_{j-1}\}$, including the empty sequence $\emptyset$ called the root, and the edges link the pairs $\{(i_1 i_2 \dots i_k), (i_1 i_2 \dots i_k i_{k+1})\}$. Note that the sequence $\bar{d}$ need not be infinite in which case the tree is finite.

The distance (each edge has length 1) from a vertex to the root is called the level of the vertex. The vertices of level $l(v) = n$ form the nth layer (or level) of cardinality $d_0 d_1 \dots d_{n-1}$.

Each vertex v of level n gives rise to a spherically homogeneous rooted subtree T_v when restricting to vertices of the form $(v i_n i_{n+1} \dots i_{n+k})$. The tree T_v is isomorphic to the tree $T_{\sigma^n \bar{d}}$ associated to the sequence $\sigma^n \bar{d} = \{d_j\}_{j\geq n}$ (with σ denoting the usual shift $\sigma : (d_0 d_1 d_2 \dots) \mapsto (d_1 d_2 d_3 \dots)$).

2.2. **Automorphism group.** An automorphism of $T_{\bar{d}}$ is a graph automorphism, that is a bijection of the set of vertices mapping edges to edges, which fixes the root. These properties imply that the layers are preserved, and an automorphism acts on a layer by permutation. The group of all such automorphisms will be denoted $Aut(T_{\bar{d}})$. Spherical homogeneity ensures that $Aut(T_{\bar{d}})$ and $Aut(T_{\sigma\bar{d}})$ are related by an isomorphism:

$$Aut(T_{\bar{d}}) \simeq Aut(T_{\sigma\bar{d}}) \wr S_{d_0}. \tag{2.1}$$

Recall that $G \wr S_d \simeq (G \times \dots \times G) \rtimes S_d$ where S_d (the group of permutation of the set $\{1, 2, \dots, d\}$) acts on the d copies of G by permutation. This identification will allow to write extensively $f = (f_1, f_2, \dots, f_{d_0})\sigma$ with f in $Aut(T_{\bar{d}})$, the f_i in $Aut(T_{\sigma\bar{d}})$ and σ in S_{d_0}. The product rule is $fg = (f_1, f_2, \dots, f_{d_0})\sigma(g_1, g_2, \dots, g_{d_0})\tau = (f_1 g_{\sigma(1)}, \dots, f_{d_0} g_{\sigma(d_0)})\sigma\tau$. In particular, there is a projection $p : Aut(T_{\bar{d}}) \to S_{d_0}$ called restriction to the first level. The kernel of this projection is called the stabilizer of the first level, denoted $St_1(Aut(T_{\bar{d}}))$, easily checked to be isomorphic to the direct product $Aut(T_{\sigma\bar{d}}) \times \dots \times Aut(T_{\sigma\bar{d}})$ with d_0 factors.

More generally for each integer n, there is an isomorphism:

$$Aut(T_{\bar{d}}) \simeq Aut(T_{\sigma^n\bar{d}}) \wr Aut(T_{d_0 \dots d_{n-1}}), \tag{2.2}$$

where $Aut(T_{d_0 \dots d_{n-1}})$ acts by permutation on $d_0 \dots d_{n-1}$ copies of $Aut(T_{\sigma^n\bar{d}})$ the way it acts on the set of leaves $\partial T_{d_0 \dots d_{n-1}}$ (the boundary) of the finite tree $T_{d_0 \dots d_{n-1}}$. There

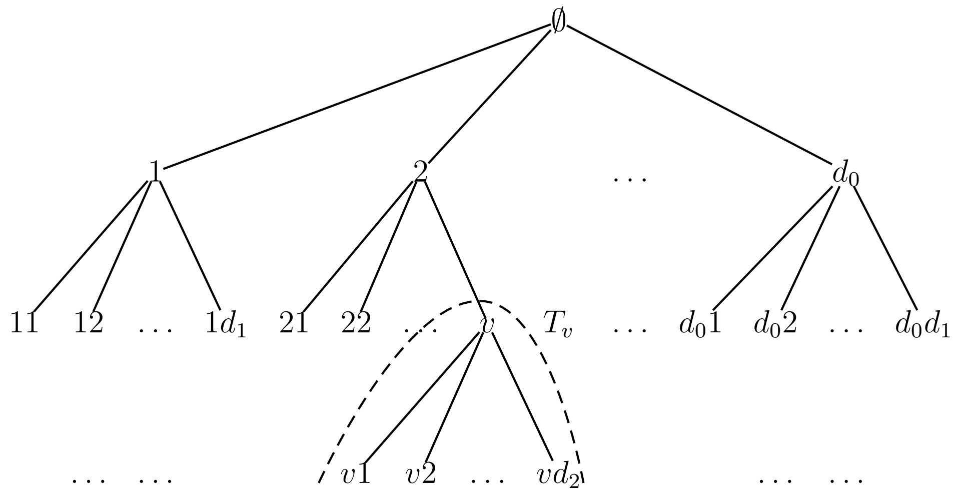

Figure 1. Spherically homogeneous rooted tree, subtree.

is also a projection $p_n : Aut(T_{\bar{d}}) \to Aut(T_{d_0 \dots d_{n-1}})$, the kernel of which constitutes the stabilizer $St_n(Aut(T_{\bar{d}}))$ of the nth level. This is a normal subgroup of $Aut(T_{\bar{d}})$ isomorphic to the direct product $St_n(Aut(T_{\bar{d}})) \simeq Aut(T_{\sigma^n \bar{d}}) \times \cdots \times Aut(T_{\sigma^n \bar{d}})$, the elements of which will occasionally be written $g = (g_{1\dots1}, \dots, g_{d_0\dots d_{n-1}})_n$.

The full group of automorphism can be viewed as a profinite group via:

$$Aut(T_{\bar{d}}) = \varprojlim_{n\to\infty} Aut(T_{d_0\dots d_{n-1}}) = \varprojlim_{n\to\infty} (S_{d_{n-1}} \wr S_{d_{n-2}} \wr \cdots \wr S_{d_0}). \tag{2.3}$$

A basis of open sets for the profinite topology associated is $\{St_n(Aut(T_{\bar{d}}))\}_{n\geq 0}$. This topology can also be defined as associated to any of the following metrics $\delta_{\bar{\lambda}}$ on $Aut(T_{\bar{d}})$. Given a decreasing sequence $\bar{\lambda} = \{\lambda_n\}_{n\geq 0}$ of positive numbers tending to zero, set:

$$\delta_{\bar{\lambda}}(g,h) = \inf\{\lambda_n | g(v) = h(v) \quad \text{for all vertices } v \text{ of level} \leq n\}.$$

A nice description of automorphisms of a rooted tree is to draw portraits. A portrait is a function g from the set of all vertices v of the tree $T_{\bar{d}}$ taking permutation values $g(v) \in S_{d_{l(v)}}$. A portrait gives rise to a unique automorphism via the formula:

$$g(i_1 i_2 i_3 \dots i_k) = (g(\emptyset)i_1)(g(i_1)i_2)(g(i_1 i_2)i_3)\dots(g(i_1\dots i_{k-1})i_k).$$

Conversely, every automorphism has a unique portrait. The metrics $\delta_{\bar{\lambda}}$ are such that two automorphisms are n-close if their portraits coincide on the n first layers.

An automorphism is said to be even (or alternate) if all the permutations $g(v) \in S_{d_{l(v)}}$ involved in the portrait are alternate permutations $g(v) \in \mathcal{A}_{d_{l(v)}}$. The group of alternate automorphisms will be denoted $Aut^e(T_{\bar{d}})$. It satisfies:

$$Aut^e(T_{\bar{d}}) = \varprojlim_{n\to\infty} Aut^e(T_{d_0\dots d_{n-1}}) = \varprojlim_{n\to\infty} \mathcal{A}_{d_{n-1}} \wr \mathcal{A}_{d_{n-2}} \wr \cdots \wr \mathcal{A}_{d_0}, \tag{2.4}$$

the profinite topology, the distances associated and the stabilizers of levels are defined in the same way as for the full automorphism group. Note that if T_2 is a 2-regular rooted tree, then $Aut^e(T_2)$ is the trivial group.

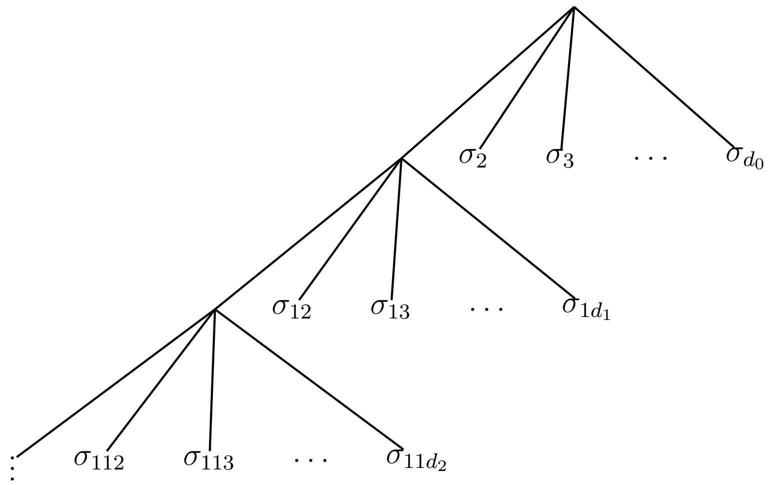

FIGURE 2. The ι-action of $\bar{H}$.

2.3. **Directed automorphism subgroups.** This paper focuses on specific subgroups of $Aut(T_{\bar{d}})$, those directed by a given infinite geodesic of the tree $T_{\bar{d}}$ starting from the root. Such a geodesic can always be chosen to be that passing at all vertices indexed by $11\dots1$ (the leftmost geodesic in the illustrations). First introduce actions of some permutation groups on $T_{\bar{d}}$. The group S_{d_0} acts on the rooted tree by permuting the subtrees of the first layer:

$$\iota_0 : S_{d_0} \hookrightarrow Aut(T_{\bar{d}}).$$

More precisely, ι_0 is defined by $\iota_0(\sigma)(i_1 i_2 \dots i_k) = \sigma(i_1) i_2 \dots i_k$. For simplicity of notation, we will identify $\sigma = \iota_0(\sigma) = (id_{T_{\sigma\bar{d}}}, \dots, id_{T_{\sigma\bar{d}}})\sigma$ and call those rooted automorphisms (their portrait is trivial outside of the root).

The infinite direct product $\bar{H} = S_{d_1} \times \dots \times S_{d_1} \times S_{d_2} \times \dots \times S_{d_2} \times \dots$ of permutation groups where S_{d_k} appears $d_{k-1}-1$ times also acts in a canonical way (once a geodesic is chosen) on the rooted tree $T_{\bar{d}}$:

$$\iota : \bar{H} \hookrightarrow Aut(T_{\bar{d}}).$$

Indeed, consider the vertices $1_k i = 1\dots1i$ with k ones and i in $\{2, \dots, d_k\}$. They form the set P of vertices at distance exactly 1 of the leftmost geodesic $111\dots$. Each permutation group S_{d_k} acts on a subtree $T_{1_k i}$ via the above homomorphism ι_0 (corresponding to the rooted tree $T_{\sigma^k \bar{d}}$). More precisely, the action is recursively defined through the wreath product by:

$$\iota(\sigma_2, \dots, \sigma_{d_0}, \sigma_{12}, \dots, \sigma_{1d_1}, \dots) = (\iota'(\sigma_{12}, \dots, \sigma_{1d_1}, \dots), \sigma_2, \dots, \sigma_{d_0}),$$

where $\iota'(\sigma_{12}, \dots, \sigma_{1d_1}, \dots)$ represents the action of the restriction $\bar{H} \twoheadrightarrow \bar{H}_1$ via:

$$\iota' : \bar{H}_1 = S_{d_2}^{d_1-1} \times S_{d_3}^{d_2-1} \times \dots \hookrightarrow Aut(T_1) \simeq Aut(T_{\sigma\bar{d}})$$

The geometry of the set P ensures that the action of different factors commute, thus ι is a well defined injection. This is best understood by Figure 2, showing the portrait is non trivial only on P. The automorphisms obtained in $\iota(\bar{H})$ are said to be directed by the geodesic $111\dots$.

Given a subgroup A of S_{d_0} and a subgroup H of $\bar{H}$, denote by $G(A,H)$ the subgroup of $Aut(T_{\bar{d}})$ generated by $\iota_0(A)$ and $\iota(H)$. Such a group will be called a directed group of automorphisms. Note that the group H might not be countable as $\bar{H}$ is not. The group $G(S_{d_0},\bar{H})$ will be called full group of directed automorphisms. Note that the isomorphism (2.1) induces an isomorphism:

$$G(S_{d_0},\bar{H}) \simeq G(S_{d_1},\bar{H}_1) \wr S_{d_0}. \tag{2.5}$$

The class of groups of the form $G(A,H)$ has been considered in [Gri2]. It gathers many famous examples such as the family of Aleshin-Grigorchuk groups known to be torsion (see [Ale1]) and of intermediate growth (see [Gri1]). Other interesting examples are some groups of non uniform growth constructed by Wilson ([Wil1], [Wil2]) and Bartholdi ([Bar]), to which Section 7 is devoted.

3. The main theorem

In this section the Main theorem on full directed automorphism groups is stated and its proof is reduced to the proof of the a priori weaker Theorem 3.2.

Theorem 3.1 (Main Theorem). *Let $\bar{d} = (d_i)_{i\geq 0}$ be a sequence of integers $d_i \geq 2$, let S_{d_0}, $\bar{H}$ and $G(S_{d_0},\bar{H})$ be the full directed subgroup of $Aut(T_{\bar{d}})$, then:*

1) *if the sequence $\bar{d}$ is bounded, the group $G(S_{d_0},\bar{H})$ is amenable.*
2) *if the sequence $\bar{d}$ is unbounded, the group $G(S_{d_0},\bar{H})$ contains a free group $\mathbb{F}_2$ on two generators.*

The proof of part 1) of the Main Theorem 3.1 reduces to proving the following, which will be the object of Sections 4 and 5.

Theorem 3.2. *Let $\bar{d} = (d_i)_{i\geq 0}$ be a bounded sequence of integers $2 \leq d_i \leq D$, let $H < \bar{H}$ be a finite saturated subgroup, then the directed subgroup $G(S_{d_0},H)$ of $Aut(T_{\bar{d}})$ is amenable.*

Proof of part 2). The second part of the Main Theorem is an immediate consequence of the following lemma stated in [Wil2] (see also [TW]).

Lemma 3.3 ([Wil2]). *Let F be the free product of two non-trivial finite groups which are not both of order 2, and $\mathcal{S}$ be any infinite subset of $\mathbb{N}$. Then the alternate permutation group $\mathcal{A}_d$ is a homomorphic image of F for all sufficiently large d and the intersection of the kernels of all epimorphisms from F to groups $\mathcal{A}_d$ with $d \in \mathcal{S}$ is the trivial subgroup.*

This implies that if $\bar{d}$ is unbounded then the group $\bar{H}$ already contains a free group $\mathbb{F}_2$ on two generators. Indeed, let $F = \mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z}$ the free group generated by elements x of order 2 and y of order 3. Let D be such that there is an onto homomorphism $\varphi_d : F \to \mathcal{A}_d$ when $d \geq D$. Define:

$$h_1 = (\varphi_{d_1}(x), \dots, \varphi_{d_1}(x), \varphi_{d_2}(x), \dots, \varphi_{d_2}(x), \dots) \in \bar{H}$$
$$h_2 = (\varphi_{d_1}(y), \dots, \varphi_{d_1}(y), \varphi_{d_2}(y), \dots, \varphi_{d_2}(y), \dots) \in \bar{H}$$

where $\varphi_d(x) = \varphi_d(y) = 1 \in \mathcal{A}_d \subset S_d$ if $d < D$. Then Lemma 3.3 ensures that the subgroup $\langle h_1, h_2 \rangle < \bar{H}$ is isomorphic to F which contains $\mathbb{F}_2$ as a subgroup of finite index. $\square$

If the sequence $\bar{d}$ is bounded then the properties of the group $\bar{H}$ are much different.

Fact 3.4. *Let $\bar{H} = T_1 \times T_2 \times \dots$ where the groups T_i belong to a finite family $\mathcal{F} = \{F_1, \dots F_D\}$ of finite groups, then every finitely generated subgroup H' of $\bar{H}$ is finite.*

Proof. Let $h_1, \dots, h_k$ be generators of H', they are of the form $h_j = (h_j^1, h_j^2, \dots)$ with $h_j^i \in T_i$. There are at most $M = D.(max\{\#F_i\})^k$ different $k+1$-tuples $(h_1^i, h_2^i, \dots, h_k^i, F_i)$. Let I be a subset of $\mathbb{N}$ of size less than M such that all different $(k+1)$-tuples appear when i describes I. Then the projection $\pi_I : \bar{H} \to \times_{i \in I} T_i$ is injective, so that H' is finite. $\square$

Definition 3.5. A finite subgroup H of the group $\bar{H} = T_1 \times T_2 \times \dots$ where the T_i belong to a finite family $\mathcal{F}$ of finite group is said to be *saturated* if the equidistributed probability measure q_H on H projects on each coordinate i to the equidistributed probability measure q_{T_i} on T_i, that is if $h = (h_1, h_2, \dots) \in H$ then $q_H(h_i = t) = q_{T_i}(t) = \frac{1}{\#T_i}$.

Fact 3.6. *Every finite subgroup H' of $\bar{H}$ is included in a finite saturated group H.*

Proof. With the above notations set for each i in I:

$$J_i = \{j \in \mathbb{N} | (h_1^j, h_2^j, \dots, h_k^j, F_j) = (h_1^i, h_2^i, \dots, h_k^i, F_i)\}.$$

There is a diagonal embedding $T_i \to \times_{j \in J_i} T_j$ and as $\cup_{i \in I} J_i = \mathbb{N}$ we get a diagonal injection:

$$\times_{i \in I} T_i \hookrightarrow \bar{H}$$

the image H of which contains H' and is saturated by construction, knowing a finite direct product is always saturated. $\square$

Proof that Theorem 3.2 implies the Main Theorem. To prove the group $G(S_{d_0}, \bar{H})$ is amenable, it is sufficient to prove amenability for every finitely generated subgroup G_f (Theorem 1.2.7. in [Gre]), which reduces, assuming Theorem 3.2, to show that G_f is included in some $G(S_{d_0}, H)$ for H finite saturated. Indeed, let $s_1, \dots, s_k$ be generators of G_f, each s_j is of the form $s_j = a_j^1 h_j^2 a_j^3 \dots h_j^{n_j}$, with $a_j^i \in S_{d_0}$ and $\langle (h_j^i)_{i,j} \rangle < \bar{H}$ finitely generate a subgroup H' which is included in some finite saturated subgroup H by Facts 3.4 and 3.6. $\square$

4. Scheme of the proof of Theorem 3.2

This section is devoted to the scheme of the proof of Theorem 3.2 which implies the Main Theorem 3.1. The details are given in Section 5. Groups of the form $G(S_{d_0}, H)$ share similarities with the Basilica group defined by a three state automaton introduced by Grigorchuk and Zuk in [GZ]. The Basilica group was shown to be amenable by Bartholdi and Virag (see [BV]) using selfsimilarity of some random walks. This method, called the "Münchhausen trick", has been used to show amenability of a few other groups (see [Kai] and [Muc]). We proceed with the same methods, using Kesten's criterion on symmetric random walks.

As H is a finite saturated subgroup of $\bar{H} = S_{d_1}^{d_0-1} \times S_{d_2}^{d_1-1} \times \dots$, let us denote H_k its restriction to $\bar{H}_k = S_{d_{k+1}}^{d_k-1} \times S_{d_{k+2}}^{d_{k+1}-1} \times \dots$ which is also a finite saturated subgroup and it follows from (2.5) that $G(S_{d_0}, H) \hookrightarrow G(S_{d_1}, H_1) \wr S_{d_0}$, and more generally the group $G(S_{d_k}, H_k)$ is a directed subgroup of $Aut(T_{\sigma^k \bar{d}})$ satisfying the crucial:

$$G(S_{d_k}, H_k) \hookrightarrow G(S_{d_{k+1}}, H_{k+1}) \wr S_{d_k}.$$

The word metric does not behave appropriately enough through this wreath product embedding, rather use:

Proposition 4.1 (A fractal family of pseudo norms of exponential growth)**.** *There exists a family of pseudo norms ν^k on $G(S_{d_k}, H_k)$ (which means symmetric positive functions $\nu^k : G(S_{d_k}, H_k) \to \mathbb{R}^+$ satisfying the triangle inequality) such that:*

a) *if g belongs to $G(S_{d_k}, H_k)$ and has image $g = (g_1, \dots, g_{d_k})\sigma$ in $G(S_{d_{k+1}}, H_{k+1}) \wr S_{d_k}$, then $\nu^k(g) \leq \nu^{k+1}(g_1) + \dots + \nu^{k+1}(g_{d_k})$, and*

b) *if $B_{\nu^k}(r) = \{g \in G(S_{d_k}, H_k) | \nu^k(g) \leq r\}$, then $\# B_{\nu^k}(r) \leq C^r$ where C is a constant depending only on the bound D on the valencies of the tree and the size of the finite group H (which contains H_k for every k).*

Let p denote the symmetric probability measure on the finite generating set $S_{d_0} \cup H$ of $G(S_{d_0}, H)$ defined by $p(a) = \frac{1}{2\# S_{d_0}}$ for $a \in S_{d_0}$ and $p(h) = \frac{1}{2\# H}$ for $h \in H$. The random walk associated is $Z_N = s_1 \dots s_N$ where the s_i are independent random variables identically p-distributed. The set of random sequences $(Z_N)_{N \in \mathbb{N}}$ is endowed with the product measure (defined on the sigma algebra generated by cylinders) $P = p^{\otimes \infty}$. The drift of this random walk with respect to the pseudo norm $\nu = \nu^0$ vanishes:

Proposition 4.2. *The random walk (Z_N) satisfies:*

$$\frac{\nu(Z_N)}{N} \underset{N \to +\infty}{\longrightarrow} 0, \quad P \text{ a.s..}$$

To prove this proposition, another (non symmetric) random walk is usefull. Let us define $Y_n = t_0 t_1 t_2 \dots t_n$ where t_{2i} are random variables equidistributed on S_{d_0} and t_{2i+1} are equidistributed on H and all the t_i are independent. Denote $Q = (q_{S_{d_0}} \otimes q_H)^{\otimes \infty}$ the associated measure on the set of sequences $(Y_n)_{n \in \mathbb{N}}$ (with respect to the cylindrical sigma algebra), then:

Proposition 4.3. *The random walk (Y_n) satisfies:*

$$\frac{\nu(Y_n)}{n} \underset{n \to +\infty}{\longrightarrow} 0, \quad Q \text{ a.s..}$$

The key argument to prove Proposition 4.3 is the next Lemma 4.6 together with Proposition 4.1 a).

Remark 4.4 (On the dependence on t_0). The pseudo norm $\nu = \nu^0$ satisfies $\nu(ah) = \nu(h)$ for every a in S_{d_0} and h in H (Proposition 5.2 (2)), which ensures $\nu(Y_n) =$

$\nu(t_0^{-1}Y_n) = \nu(t_1t_2\dots t_n)$, showing that $\nu(Y_n)$ is independent of t_0. This will be of importance and justifies the:

Definition 4.5. Two random variables U and V on $G(S_{d_0}, H)$ are said ν-equivalent if $\nu(U)$ and $\nu(V)$ have the same distribution law on $\mathbb{N}$, which will be denoted:

$$U \sim_{\nu\text{-law}} V.$$

Consider the random walk $(Y_n)_{n\in\mathbb{N}}$ and its image in the wreath product of the form $Y_n = (Y_n^1, \dots, Y_n^{d_0})\sigma_n$ where σ_n is a random variable in S_{d_0} and the coordinates Y_n^t for $t \in \{1, \dots, d_0\}$ are random variables in $G(S_{d_1}, H_1)$. The point is that $(Y_n^t)_n$ follows the law of the similarly defined random walk $(Y'_m)_{m\in\mathbb{N}}$ on $G(S_{d_1}, H_1)$ (which is taking independent equidistributed increments alternatively in S_{d_1} and H_1), but at a slower speed. More precisely:

Lemma 4.6 (Similarity of the random walks (Y_n) and (Y'_m))**.** *Let $(Y_n)_{n\in\mathbb{N}}$ the random walk defined above and $Y_n = (Y_n^1, \dots, Y_n^{d_0})\sigma_n$ its image in the wreath product. For each coordinate $(Y_n^t)_n$ the sequence $(Y_n)_n$ defines a sequence of random integers $(m_t(n))_n$ and a random sequence $(\varepsilon_t(n))_n$ taking values in $\{0,1\}$ such that:*

1. *For every integer n the values of $m_t(n)$ and $\varepsilon_t(n)$ depend only on $(Y_{n'})_{n'\leq n}$.*
2. *For every integer n the coordinate Y_n^t belonging to $G(S_{d_1}, H_1)$ has the same ν^1-distribution law as the random variable $Y'_{m_t(n)+\varepsilon_t(n)}$. More precisely the conditional law:*
$$(Y_n^t | m_t(n), \varepsilon_t(n)) \sim_{\nu^1\text{-}law} Y'_{m_t(n)+\varepsilon_t(n)}.$$
3. *The random sequence $(m_t(n))_n$ satisfies:*
$$m_t(n) \sim_{n\to+\infty} \left(\frac{d_0-1}{d_0}\right)\frac{n}{d_0}, \quad Q \text{ a.s.}.$$

Propositions 4.1 and 4.2 are sufficient to apply the:

Theorem 4.7 (Kesten criterion of amenability [Kes])**.** *Let Γ be a finitely generated group and (Z_N) a symmetric random walk on Γ. The group Γ is amenable if and only if the sequence $(P(Z_{2N} = id_\Gamma))_N$ does not decay exponentially fast with N.*

The following fact is also usefull:

Fact 4.8. *Let (Z_N) a symmetric random walk on a finitely generated group Γ, then for any fixed integer N the function $\Gamma \to [0,1] : g \mapsto P(Z_{2N} = g)$ is maximal for $g = id_\Gamma$.*

Proof of the Fact 4.8. Let $p_k(x, y)$ denote the probability to go from x to y in k steps, let δ_x denote the function on Γ taking values 1 on x and 0 elsewhere and M the symmetric random walk operator on the space $l^2(\Gamma)$. Then Cauchy inequality implies:

$$\begin{aligned} p_{2N}(id, x)^2 &= \langle M^{2N}\delta_{id}, \delta_x\rangle^2 = \langle M^N\delta_{id}, M^N\delta_x\rangle^2 \\ &\leq ||M^N\delta_{id}||.||M^N\delta_x|| = p_{2N}(id, id).p_{2N}(x, x) = p_{2N}(id, id)^2. \end{aligned}$$

□

Note that Theorem 4.7 and Fact 4.8 only apply to symmetric random walks.

Proof of Theorem 3.2. Given an arbitrary positive ε the previous Fact 4.8 applied to the symmetric random walk (Z_N) constructed above raises:

$$P(\nu(Z_{2N}) \leq \varepsilon 2N) = \sum_{\nu(g)\leq\varepsilon 2N} P(Z_{2N} = g) \leq P(Z_{2N} = id_{G(S_{d_0},H)})\#B_\nu(\varepsilon 2N),$$

and the Propositions 4.1 b) and 4.2 ensure:

$$P(Z_{2N} = id) \geq P\left(\frac{\nu(Z_{2N})}{2N} \leq \varepsilon\right) C^{-\varepsilon 2N} \sim_{N\to\infty} C^{-\varepsilon 2N}.$$

Thus $P(Z_{2N} = id)$ does not decrease exponentially fast and Kesten's criterion proves Theorem 3.2 and thus the Main Theorem. □

5. Details of the proof of Theorem 3.2

5.1. **Fractal pseudo norms of exponential growth (proof of Proposition 4.1).** To the symmetric generating set $S = (S_{d_0} \cup H)\backslash\{1\}$ of $G(S_{d_0}, H)$ is associated the word norm on $G(S_{d_0}, H)$ by:

$$|g| = \min\{r | g = z_1 \dots z_r, z_i \in S\}.$$

Denote $B_S(r)$ the ball of radius r associated to this norm (that is the set of all g such that $|g| \leq r$), then $\#B_S(r) \leq (\#S)^r$.

Note that since $G(S_{d_0}, H)$ is a quotient of the free product $S_{d_0} * H$ a word $z_1 \dots z_r$ is a minimal representative of g (that is $r = |g|$) only in the following cases: either $z_{2j} \in S_{d_0}\backslash\{1\}$ and $z_{2j+1} \in H\backslash\{1\}$, or conversely. This brings another definition:

$$||g||_0 = \min\{r | g = a_1 h_1 a_2 h_2 \dots h_r a_{r+1}, a_i \in S_{d_0}, h_j \in H\}. \tag{5.1}$$

The following is straightforward:

Properties 5.1. *The function $||.||_0$ is a norm when restricted to the stabilizer of the first level $St_1(G(S_{d_0}, H))$, namely it satisfies:*

(1) $||gh||_0 \leq ||g||_0 + ||h||_0$ for all g, h in $G(S_{d_0}, H)$,
(2) $||g^{-1}||_0 = ||g||_0$ for all g in $G(S_{d_0}, H)$,
(3) $||g||_0 = 0$ if and only if $g \in S_{d_0}$,

This function $||.||_0$ is related to the usual word norm since for g in $G(S_{d_0}, H)$:

$$2||g||_0 - 1 \leq |g| \leq 2||g||_0 + 1,$$

which implies that if $B_{||.||_0}(r)$ is the ball of radius r associated to $||.||_0$ in $G(S_{d_0}, H)$, then:

$$\#B_{||.||_0}(r) \leq (\#S)^{2r+1}.$$

Following [BV], let us introduce a new function ν on $G(S_{d_0}, H)$ which is to be thought of as a fractal distance. For $g \in G(S_{d_0}, H)$ and a vertex v on layer $k = l(v)$ of $T_{\bar{d}}$, denote by g_v the action of g on the descendant subtree $T_v \simeq T_{\sigma^k \bar{d}}$ of $T_{\bar{d}}$ and $g(v) \in S_{d_k}$ the action on the d_k children of v. The automorphism g_v of the rooted tree T_v belongs to the group $G(S_{d_k}, H_k)$. The function defined by (5.1) for $G(S_{d_k}, H_k)$ will be denoted by $||.||_k$.

A subtree T of $T_{\bar{d}}$ is said to be rooted if it contains the root $\emptyset$ of $T_{\bar{d}}$. It is said regular if for every vertex $v \in T$, either T contains the $d_{l(n)}$ descendant of v, either it contains none of them.

Given a finite regular rooted subtree T of $T_{\bar{d}}$ with set of leaves ∂T, define a function ν_T on $G(S_{d_0}, H)$ by:

$$\nu_T(g) = \sum_{v \in \partial T} (1 + ||g_v||_{l(v)}).$$

and a function $\nu : G(S_{d_0}, H) \to \mathbb{N}$ as:

$$\nu(g) = \min\{\nu_T(g) | T \text{ is a finite regular rooted subtree of } T_{\bar{d}}\}. \tag{5.2}$$

The construction (5.2) defines similarly a function $\nu^k : G(S_{d_k}, H_k) \to \mathbb{N}$ for the subgroup $G(S_{d_k}, H_k) < Aut(T_{\sigma^k \bar{d}}) \simeq Aut(T_v)$ for any vertex v on the kth layer. Note that $\nu = \nu^0$ and that the following proposition is still true replacing ν by ν^k and ν^1 by ν^{k+1}.

Proposition 5.2. *The function ν satisfies:*

(1) Let g in $G(S_{d_0}, H)$ and $g = (g_1, \ldots, g_{d_0})\sigma$ be its embedded image in the wreath product $G(S_{d_0}, H) \hookrightarrow G(S_{d_1}, H_1) \wr S_{d_0}$, then:

$$\nu(g) = \min\{\nu^1(g_1) + \cdots + \nu^1(g_{d_0}), 1 + ||g||_0\}.$$

(2) Let g in $G(S_{d_0}, H)$, then $\nu(g) = \nu(g^{-1})$.
(3) Let g, g' be in $G(S_{d_0}, H)$, then $\nu(gg') \leq ||g||_0 + \nu(g')$.
(4) Let g, g' be in $G(S_{d_0}, H)$, then $\nu(gg') \leq \nu(g) + \nu(g')$.

In particular, this function ν is a pseudo-norm on $G(S_{d_0}, H)$.

The use of induction in the proof of Proposition 5.2 requires the:

Property 5.3. *Let g in $G(S_{d_k}, H_k)$ have image $g = (g_1, \ldots, g_{d_k})\sigma$ in the wreath product $G(S_{d_{k+1}}, H_{k+1}) \wr S_{d_k}$ and assume $||g||_k \geq 2$ then $||g_t||_{k+1} < ||g||_k$ for any coordinate t.*

Proof of Property 5.3. An element g admits a minimal representative of the form $g = h_1^{\sigma_1} \ldots h_r^{\sigma_r} \sigma_{r+1}$ with σ_i in S_{d_k} and h_i in H_k (remind $x^y = yxy^{-1}$). Moreover by construction $h = (h', a_2, \ldots, a_{d_k})$ with h' in H_{k+1} and a_i in $S_{d_{k+1}}$ and the conjugate h^σ is the same d_k-tuple where the coordinates are σ permuted. This ensures $||g_1||_{k+1} + \cdots + ||g_{d_0}||_{k+1} \leq ||g||_k$. It is sufficient to prove the property for $||g||_k = 2$, that is $g = h_1^{\sigma_1} h_2^{\sigma_2}$. If $\sigma_1(1) \neq \sigma_2(1)$ the property is obvious. If $\sigma_1(1) = \sigma_2(1)$ then $||g_i||_{k+1} = 0$ if $i \neq \sigma_1(1)$ and $||g_{\sigma_1(1)}||_{k+1} = ||h_1' h_2'||_{k+1} = 1$ because $h_1' h_2'$ is an element of H_{k+1}. $\square$

Proof of Proposition 5.2. Note that $1 + ||g||_0 = \nu_{\{\emptyset\}}(g)$ and assume the minimum in definition (5.2) is obtained for a finite regular rooted tree $T \neq \{\emptyset\}$. Clearly

$\partial T = \partial T(1) \cup \cdots \cup \partial T(d_0)$ where $T(v)$ denotes the intersection of T with the subtree T_v of $T_{\bar{d}}$ hung on vertex v, thus:

$$\nu_T(g) = \sum_{v \in \partial T} (1 + ||g_v||_{l(v)}) = \sum_{t=1}^{d_0} \sum_{v \in \partial T(t)} (1 + ||g_v||_{l(v)}) = \sum_{t=1}^{d_0} \nu_{T(t)}(g_t),$$

which is minimal if and only if $\nu_{T(t)}(g_t) = \nu^1(g_t)$ is minimal for all t. This implies part (1).

It follows that if $||g||_0 = 1$ then $\nu(g) = 2 = \nu(g^{-1})$. Similarly if $||g||_k = 1$ for $g \in G(S_{d_k}, H_k)$ then $\nu^k(g) = 2 = \nu^k(g^{-1})$. Assume by induction on r that $\nu^k(g) = \nu^k(g^{-1})$ if $||g||_k \leq r$ and this jointly for every level k, then the inverse formula $g^{-1} = \sigma^{-1}(g_1^{-1}, \ldots, g_{d_0}^{-1})$ and the induction hypothesis ensuring $\nu^1((g^{-1})_1) + \cdots + \nu^1((g^{-1})_{d_0}) = \nu^1(g_1) + \cdots + \nu^1(g_{d_0})$ (as $||g_t||_{k+1} < ||g||_k$ by Property 5.3) together with part (1) show part (2).

To prove part (3), note first that $\nu(ag) = \nu(g)$ for all $a \in S_{d_0}$. Indeed, a only permutes the subtrees of the first level and does not increase any of the $||g_v||_{l(v)}$. To conclude, it is sufficient to show that when h is in H, we have $\nu_T(hg) \leq 1 + \nu_T(g)$ for any finite regular subtree T. Proceed by induction on the size of T. Indeed, this is true for $T = \{\emptyset\}$ by Property 5.1 (1). More generally, denoting $g = (g_1, \ldots, g_{d_0})\sigma_0$ and $h = (h_1, a_2, \ldots, a_{d_0})$ with g_t in $G(S_{d_1}, H_1)$, h_1 in H_1 and a_t in S_{d_1}, we get $hg = (h_1 g_1, a_2 g_2, \ldots, a_{d_0} g_{d_0})\sigma_0$ and:

$$\nu_T(hg) = \nu_{T(1)}(h_1 g_1) + \sum_{t=2}^{d_0} \nu_{T(t)}(a_t g_t) \leq 1 + \nu_{T(1)}(g_1) + \sum_{t=2}^{d_0} \nu_{T(t)}(g_t) = 1 + \nu_T(g)$$

using the induction hypothesis on $T(1)$.

Part (4) is implied by part (3) in case $\nu(g) = 1 + ||g||_0$ or $\nu(g') = 1 + ||g'||_0$. Otherwise:

$$\nu(gg') \leq \sum_{t=1}^{d_0} \nu^1((gg')_t) = \sum_{t=1}^{d_0} \nu^1(g_t g'_{\sigma(t)}) \leq \sum_{t=1}^{d_0} \nu^1(g_t) + \nu^1(g_{\sigma(t)}) = \nu(g) + \nu(g'),$$

where the second inequality comes by joint induction on $||g||_k$ using Property 5.3. □

Let $B_\nu(r) = \{g \in G(S_{d_0}, H) | \nu(g) \leq r\}$ denote the ball of radius r associated to the function ν. The next proposition is crucial for our purpose.

Proposition 5.4. *Consider a spherically homogeneous rooted tree $T_{\bar{d}}$ of bounded valency $2 \leq d_i \leq D$, a finite subgroup H of $\bar{H}$ and the function ν constructed above, then the balls $B_\nu(r) \subset G(S_{d_0}, H)$ grow at most exponentially fast. Namely, there exists a constant C depending only on D and the size of H such that:*

$$\#B_\nu(r) \leq (C)^r, \quad \textit{for all } r \textit{ sufficiently large.}$$

In order to prove this proposition, recall classical estimates on the number of rooted subtrees of a rooted tree. The formula below can be found in [PR], the equivalent is derived from Stirling's formula.

Proposition 5.5. *The number of (not necessarily regular) rooted subtrees of a D-regular tree T_D containing r vertices is:*

$$s_r^{(D)} = \frac{1}{r} C_{Dr}^{r-1} \sim_{r \to +\infty} \frac{1}{D-1} \sqrt{\frac{D}{2(D-1)\pi}} r^{-\frac{3}{2}} \left(\frac{D^D}{(D-1)^{(D-1)}} \right)^r.$$

More precisely the following is sufficient:

Corollary 5.6. *The number $t_r^{(D)}$ of regular rooted subtrees of $T_{\bar{d}}$ (with $\bar{d}$ bounded by D) containing at most r leaves satisfies:*

$$t_r^{(D)} \leq (K_D)^r, \qquad \textit{for } K_D = \frac{D^{2D}}{(D-1)^{2(D-1)}},$$

provided r is sufficiently large.

Proof. It is well known that a subtree with at most r leaves contains at most $2r-1$ vertices and the asymptotic equivalent of $s_r^{(D)}$ gives the corollary. □

Proof of Proposition 5.4. If $\nu(g) \leq r$ then there exists a regular rooted subtree T such that $\nu_T(g) \leq r$. In particular, such a subtree has less than r leaves so that there are at most $(K_D)^r$ choices for T (corollary 5.6). Given T, the element g is described by all $g(v) \in S_{d_{l(v)}}$ where $v \in \mathring{T}$, which allow at most $(D!)^{\#\mathring{T}} \leq (D!)^r$ choices, and all $g_v \in G(S_{d_{l(v)}}, H_{l(v)})$ with $v \in \partial T$, which satisfy:

$$\sum_{v \in \partial T} ||g_v||_{l(v)} \leq r.$$

The number of possibilities for this last choice is less than $(M+1)^{2r}$ where $M = max\{\#B_{||\cdot||_k}(1)\}$ (finite because the size of the generating set $S_{d_k} \cup H_k$ on layer k depends only on $d_k \leq D$ and $\#H_k \leq \#H$) bounds the number of symbols which represent an automorphism of norm 1 on a given leaf. An extra symbol (a coma) is added to denote passing to the next leaf. All in all, taking $C = K_D D!(M+1)^2$ gives the desired result. □

5.2. **Similarity of random walks (proof of Lemma 4.6).** First recall elementary probabilistic facts which will be usefull.

Fact 5.7. *Let $(z_i)_{i \geq 1}$ be independent random variables equidistributed on a finite group F. Then the sequence $(X_k)_{k \geq 1}$ of products $X_k = z_1 \dots z_k$ is a family of independent random variables equidistributed on F.*

Proof of Fact 5.7. Denote by q_F the equidistribution measure on the finite group F. It is sufficient to prove by induction that:

$$q_F^{\otimes \infty}(X_i = f_i, i \leq k) = \prod_{i=1}^{k} q_F^{\otimes \infty}(X_i = f_i) = \prod_{i=1}^{k} q_F(X_i = f_i),$$

for arbitrary $f_1, \dots, f_k$ in F, which comes from:

$$\begin{aligned} q_F^{\otimes\infty}(X_i = f_i, i \leq k) &= q_F^{\otimes\infty}(X_k = f_k | X_i = f_i, i \leq k-1) q_F^{\otimes\infty}(X_i = f_i, i \leq k-1) \\ &= q_F^{\otimes\infty}(z_k = f_{k-1}^{-1} f_k) \prod_{j=1}^{k-1} q_F^{\otimes\infty}(X_j = f_j) \\ &= q_F(z_k = f_{k-1}^{-1} f_k) \prod_{j=1}^{k-1} q_F(X_j = f_j). \end{aligned}$$

□

Fact 5.8. *Let z be a random variable equidistributed on a finite group F acting transitively on a finite set A, then $q_F(z(t) = t') = \frac{1}{\#A}$ for all t, t' in A.*

Proof of Fact 5.8. The quotient $F/Stab_F(t)$ is of size $\#A$. If $z_0(t) = t'$ (transitivity) then $z_0 Stab_F(t) = \{z | z(t) = t'\}$ has the same size as $Stab_F(t)$ by injectivity of left translation in F. □

Fact 5.9. *Let $(u_i)_{i \in \mathbb{N}}$ be independent random Bernoulli variables on $\{0, 1\}$ (say $p(u_i = 0) = p$ and $p(u_i = 1) = 1 - p$ for some p in $]0, 1[$). Let $f(w_N)$ be the number of alternations in the subsequence $w_N = u_1 \dots u_N$, that is the number of indexes i such that $u_i \neq u_{i+1}$. Equivalently, $1 + f(w_N)$ is the number of maximal packs of constant successive terms. Then:*

$$f(w_N) \sim_{N \to +\infty} 2p(1-p)N, \qquad P = p^{\otimes\infty} \ a.s..$$

Proof of Fact 5.9. Apply the law of large numbers to $f(w_N) = \sum_{i=1}^{N-1} 1_{\{u_i \neq u_{i+1}\}}$ knowing that $E(1_{\{u_i \neq u_{i+1}\}}) = 2p(1-p)$ and that the terms are independent. □

Proof of Lemma 4.6. Consider the random walk Y_n at step n as:

$$Y_n = t_0 t_1 \dots t_n = a_1 h_1 a_2 h_2 \dots a_s h_s a_{s+1}$$

with $s = [\frac{n}{2}]$ (a_{s+1} empty if n even), where the terms a_i (resp. h_i) are random variables equidistributed in S_{d_0} (resp. in H), all being independent. This can be rewritten $Y_n = h_1^{\sigma_1} \dots h_s^{\sigma_s} \sigma_{s+1}$ (remind the conjugate notation $h^\sigma = \sigma h \sigma^{-1}$) where the $\sigma_i = a_1 a_2 \dots a_i$ are independent random variables equidistributed in S_{d_0} by Fact 5.7.

Using coordinates in the wreath product an element h of H has the form $h = (h^1, a^2, \dots, a^{d_0})$ with h^1 in H_1 and a^i in S_{d_1} and each of them is equidistributed for h equidistributed in H by saturation (note that the coordinates are not independent). Conjugating by a rooted automorphism σ raises $h^\sigma = (a^{\sigma(1)}, \dots, a^{\sigma(d_0)})$ with h^1 in position $\sigma(1)$.

Consider now the random walk $Y_n = (Y_n^1, \dots, Y_n^{d_0})\sigma_n$ at time n and focus on coordinate t, which is a product $Y_n^t = u_1 \dots u_s$ of s independent terms such that u_i belongs to and is equidistributed in S_{d_1} (resp. H_1) if $\sigma_i(t)$ belongs to $\{2, \dots, d_0\}$ (resp. $\sigma_i(t) = 1$). Since the σ_i are equidistributed in S_{d_0} the probability that u_i is in S_{d_1} (resp. H_1) for a given i is $\frac{d_0-1}{d_0}$ (resp. $\frac{1}{d_0}$) by Fact 5.8. This is summarized in:

$$Q(u_i = g) = \begin{cases} \frac{d_0-1}{d_0}\frac{1}{\#S_{d_1}} & \text{if } g \in S_{d_1}, \\ \frac{1}{d_0}\frac{1}{\#H_1} & \text{if } g \in H_1, \end{cases}$$

and the terms u_i are independent. Define $m_t(n)$ to be the number of maximal packs of successive u_i belonging either to S_{d_1}, or to H_1 in the sequence $Y_n^t = u_1 \dots u_s$. Fact 5.9 ensures that:

$$m_t(n) \sim_{n\to+\infty} 2\frac{1}{d_0}\left(1-\frac{1}{d_0}\right) s \sim_{n\to+\infty} \left(\frac{d_0-1}{d_0}\right)\frac{n}{d_0}.$$

Given an integer n, assume we know the distribution $\mathcal{D}$ of which terms u_i are in S_{d_1} and H_1, then the kth pack of terms $v_k = u_{i_k}u_{i_k+1}\dots u_{j_k}$ of constant belonging is a product of equidistributed independent elements in the finite group S_{d_1} or H_1 hence is equidistributed. In this situation $Y_n^t = v_0 v_1 \dots v_{m_t(n)}$ where two cases are possible: either u_1 belongs to S_{d_1} (set $\varepsilon_t(n) = 0$), the terms v_{2k+1} are equidistributed in H_1 and v_{2k} are equidistributed in S_{d_1}, which is of the form $Y'_{m_t(n)}$; or u_1 belongs to H_1 (set $\varepsilon_t(n) = 1$), then re index the v_i as $Y_n^t = id_{S_{d_1}} v_1 \dots v_{m_t(n)+1}$ which is of the form $Y'_{m_t(n)+1}$ except for v_0 which follows the Dirac law on $id_{S_{d_1}}$; this has no influence on the ν-distribution of the sequences (Remark 4.4). In both cases:

$$(Y_n^t|\mathcal{D}) \sim_{\nu\text{-law}} Y'_{m_t(n)+\varepsilon_t(n)},$$

where the condition depends only on the number of alternations $m_t(n)$ and the starting condition $\varepsilon_t(n)$ of the distribution $\mathcal{D}$. □

5.3. **Zero drift of** (Y_n) **(proof of Proposition 4.3).** First note that the Kolmogorov 01-law implies almost sure constance of $\limsup \frac{\nu(Y_n)}{n}$.

Lemma 5.10. *For every integer k denote $(Y_n^{(k)})_n$ the random walk on $G(S_{d_k}, H_k)$ which is taking independent equidistributed increments alternatively in S_{d_k} and H_k, in particular $(Y_n) = (Y_n^{(0)})$ and $(Y'_n) = (Y_n^{(1)})$. Then there exists l_k in $[0, \frac{1}{2}]$ such that:*

$$\limsup_{n\to+\infty} \frac{\nu^k(Y_n^{(k)})}{n} = l_k, \qquad Q_k = (q_{S_{d_k}} \otimes q_{H_k})^{\otimes\infty} \text{ a.s.}.$$

Proof. Proposition 5.2 (1) implies $\nu^k(Y_n^{(k)}) \leq 1 + ||Y_n^{(k)}||_k \leq \frac{n+1}{2}$ so that the $\limsup$ is $\leq \frac{1}{2}$. Given l in $[0, \frac{1}{2}]$ the event $E_l = \{\limsup \frac{\nu^k(Y_n^{(k)})}{n} \leq l\}$ is a tail event, that is an event which is independent of any finite subsequence $(Y_n^{(k)})_{n\leq N}$, hence has probability 0 or 1 by the 01-Kolmogorov law. The function $l \mapsto Q_k(E_l)$ is increasing, right continuous and takes values in $\{0,1\}$, so that there exists l_k such that $Q_k(E_l) = 0$ for $l < l_k$ and $Q_k(E_l) = 1$ for $l \geq l_k$. Then:

$$Q_k\left(\{\limsup \frac{\nu^k(Y_n^{(k)})}{n} = l_k\}\right) = Q_k(E_{l_k} \setminus \cup_{n\geq1} E_{l_k - \frac{1}{n}}) = 1.$$

□

Proof of Proposition 4.3. To show $l_0 = 0$, prove $l_k \leq \frac{(D-1)}{D} l_{k+1}$ where D is the bound on the valencies of the spherically homogeneous rooted tree $T_{\bar{d}}$. This is sufficient as $l_k \leq \frac{1}{2}$ for every k. To ease notations, compute for $k = 0$. Proposition 4.1 (a) ensures:

$$\limsup_{n\to+\infty} \frac{\nu(Y_n)}{n} \leq \limsup_{n\to+\infty} \sum_{t=1}^{d_0} \frac{\nu^1(Y_n^t)}{n} \leq \sum_{t=1}^{d_0} \limsup_{n\to+\infty} \frac{\nu^1(Y_n^t)}{n}. \tag{5.3}$$

To compute the right side introduce the condition $(m_t(n))$:

$$\limsup_{n\to+\infty} \frac{\nu^1(Y_n^t)}{n} = \limsup_{n\to+\infty} \frac{\nu^1(Y_n^t)}{m_t(n)} \frac{m_t(n)}{n} \leq \limsup_{n\to+\infty} \frac{\nu^1(Y_n^t)}{m_t(n)} \limsup_{n\to+\infty} \frac{m_t(n)}{n},$$

where Lemma 4.6 gives $\limsup \frac{m_t(n)}{n} = (\frac{d_0-1}{d_0})\frac{1}{d_0}$, Q a.s. and

$$\limsup \frac{\nu^1(Y_n^t)}{m_t(n)} = \limsup \frac{\nu^1(Y'_{m_t(n)+\varepsilon_n^t})}{m_t(n)} = l_1, \quad Q \text{ a.s.}$$

because $m_t(n) \to +\infty$ Q a.s.. The last estimates gathered together on a Q probability one event show that:

$$l_0 \leq \sum_{t=1}^{d_0} l_1 \left(\frac{d_0-1}{d_0}\right)\frac{1}{d_0} = \left(\frac{d_0-1}{d_0}\right) l_1 \leq \left(\frac{D-1}{D}\right) l_1.$$

□

5.4. **Zero drift of (Z_N) (proof of Proposition 4.2).** Recall the:

Fact 5.11. *Let $(a_i)_{i\in\mathbb{N}}$ be a random sequence in $\{0,1\}^{\mathbb{N}}$ endowed with a probability measure μ. Assume that there exists an infinite subset I of $\mathbb{N}$ such that $\mu(a_i = 1) \geq \delta > 0$ for all $i \in I$, then $\mu(a_i = 1$ for infinitely many $i) \geq \delta$.*

Proof of Fact 5.11. Let $E = \{(a_i) | a_i = 1 \text{ infinitely often}\}$ and assume by contradiction $\mu(E) = \delta' < \delta$, this implies $\mu(E^c \cap \{a_i = 1\}) \geq \delta - \delta'$ for all i in I. However the complement of E is the infinite increasing union:

$$E^c = \cup_{n\in\mathbb{N}} \{(a_i) | a_i = 0 \text{ for } i \geq n\} = \cup_{n\in\mathbb{N}} F_n,$$

so that $\mu(F_N) \geq 1 - \frac{\delta+\delta'}{2}$ for some N. But the case $i \geq N$ raises the contradiction:

$$\mu(E^c \cap \{a_i = 1\}) = \mu((E^c \setminus F_N) \cap \{a_i = 1\}) \leq \mu(E^c \setminus F_N) \leq \frac{\delta+\delta'}{2} - \delta' = \frac{\delta-\delta'}{2}.$$

□

Proposition 4.3 will be used in the (a priori) weaker form:

Corollary 5.12. *For every positive ε and α, there exists N_0 such that for $n \geq N_0$:*

$$Q\left(\frac{\nu(Y_n)}{n} \leq \varepsilon\right) \geq 1 - \alpha.$$

Proof. Assume the statement does not hold, then there exists ε_0, α_0 and infinitely many integers n_k with $Q\left(\frac{\nu(Y_{n_k})}{n_k} \geq \varepsilon_0\right) \geq \alpha_0$ and then $Q\left(\limsup \frac{\nu(Y_n)}{n} \geq \varepsilon_0\right) \geq \alpha_0$ by Fact 5.11, contradicting Proposition 4.3. □

The random walks (Z_N) and (Y_n) are closely related by:

Fact 5.13. *Let N be a fixed integer. To each walk $Z_N = s_1 \dots s_N$ is associated the number of alternations $a(N)$ from s_i in S_{d_0} to s_{i+1} in H or vice versa. Then the conditional law of Z_N satisfies:*

$$(Z_N | a(N)) \sim_{\nu-law} Y_{a(N)}.$$

Proof. Conditioning by the distribution $\mathcal{D}$ of which terms s_i are in S_{d_0} and in H, the walk is rewritten: $Z_N = s_1 \dots s_{i_0} s_{i_0+1} \dots s_{i_1} \dots s_{i_{a(N)}} = t_0 t_1 \dots t_{a(N)}$ where $t_{2j} = s_{i_{2j-1}} \dots s_{i_{2j}}$ are equidistributed in S_{d_0} (except maybe t_0 which could be empty) and $t_{2j+1} = s_{i_{2j}} \dots s_{i_{2j+1}}$ are equidistributed in H, all factors being independent, which is the definition of the random walk $Y_{a(N)}$. The condition matters only on $a(N)$ and not $\mathcal{D}$. □

This Fact 5.13 allows us to show a weak form:

Lemma 5.14. *For every positive ε and α, there exists M such that for $N \geq M$:*

$$P\left(\frac{\nu(Z_N)}{N} \leq \varepsilon\right) \geq 1 - \alpha.$$

Proof. Fact 5.9 ensures that the conditioning term $a(N)$ satisfies $\lim_{N\to\infty} \frac{a(N)}{N} = \frac{1}{2}$, P almost surely. In particular for every positive α there exists an integer N_1 such that $P(a(N) \geq \frac{N}{3}) \geq 1 - \alpha$ for all $N \geq N_1$.

Now compute under the condition $a(N)$:

$$P\left(\frac{\nu(Z_N)}{N} \leq \varepsilon\right) = \sum_{a(N)} P\left(\frac{\nu(Z_N)}{N} \leq \varepsilon | a(N)\right) P(a(N)),$$

but if $N \geq N_1$ then $P(a(N) \leq \frac{N}{3}) \leq \alpha$. Moreover for $N \geq 3N_0$ (defined by Corollary 5.12) the condition $a(N) \geq \frac{N}{3} \geq N_0$ ensures via Fact 5.13:

$$P\left(\frac{\nu(Z_N)}{N} \leq \varepsilon | a(N)\right) = Q\left(\frac{\nu(Y_{a(N)})}{a(N)} \frac{a(N)}{N} \leq \varepsilon\right) \geq 1 - \alpha,$$

because $\frac{a(N)}{N} \leq 1$. All in all, when $N \geq \max\{N_1, 3N_0\}$:

$$P\left(\frac{\nu(Z_N)}{N} \leq \varepsilon\right) \geq \sum_{a(N) \geq \frac{N}{3}} (1-\alpha) P(a(N)) \geq (1-\alpha)^2,$$

which proves Lemma 5.14. □

The previous Lemma ensures that P almost surely: $\liminf \frac{\nu(Z_N)}{N} = 0$ (Fact 5.11). To get Proposition 4.2 use:

Theorem 5.15 (Kingman subadditive Theorem ([Kal] 9.14))**.** *Let $(X_{m,n})$ be random variables such that:*

(1) $X_{0,n} \leq X_{0,m} + X_{m,n}$ for all $0 < m < n$,
(2) $(X_{m+1,n+1})$ has the same law as $(X_{m,n})$,
(3) $E(X_{0,1}^+) < +\infty$,

then the random sequence $(\frac{X_{0,n}}{n})$ *converges almost surely.*

Applying this to $X_{n,m} = \nu(Z_m^{-1}Z_n)$ shows that the inferior limit is in fact a limit, proving Proposition 4.2. The interested reader will remark that Lemma 5.14 is sufficient for our purpose and thus the Main Theorem does not rely on Kingman's Theorem.

6. Groups of intermediate growth

6.1. **Generating pairs for alternate groups.** In his paper [Wil2] (Proposition 2.1), Wilson constructs interesting generating pairs of alternate groups $\mathcal{A}_d$:

Proposition 6.1 (Wilson [Wil2])**.** *Let $d \geq 29$, then the alternate group of permutation $\mathcal{A}_d$ of the finite set $\{1,\dots,d\}$ contains an eligible (see* [Wil2] *for the full definition) pair of elements x_d, y_d. In particular:*

1) *the pair is generating: $\langle x_d, y_d\rangle = \mathcal{A}_d$, the elements have order 2 and 3: $x_d^2 = y_d^3 = 1$, and a fixed point property that there exists α and β in $\{1,\dots,d\}$ such that: $x_d(\alpha) = y_d x_d y_d^{-1}(\alpha) = \alpha$ and $y_d(\beta) = \beta$ (up to re index we assume $\alpha = 1$ and $\beta = 2$).*
2) *let $\hat{x} = (u,1,\dots,1)x_{d_0}$ and $\hat{y} = (1,v,1,\dots,1)y_{d_0}$ belong to $Aut(T_{\bar{d}})$ with $d_0 \geq 29$ and u, v in $Aut(T_{\sigma\bar{d}})$ with $u^2 = v^3 = 1$, then the group generated by $\hat{x}$ and $\hat{y}$ contains the whole group of alternate rooted automorphisms $\mathcal{A}_{d_0}$. More precisely:*

$$\langle \hat{x}, \hat{y}\rangle \simeq \langle u, v\rangle \wr \mathcal{A}_{d_0}.$$

Given a (not necessarily bounded) sequence $\bar{d}$ of integers ≥ 29, the above Proposition 6.1 allows to define recursively the following pair of automorphisms of the spherically homogeneous rooted tree $T_{\bar{d}}$ (remind the assumption on fixed points $\alpha = 1$ and $\beta = 2$):

$$\begin{aligned} x_{\bar{d}} &= (x_{\sigma\bar{d}}, 1, \dots, 1)x_{d_0}, \\ y_{\bar{d}} &= (1, y_{\sigma\bar{d}}, 1, \dots, 1)y_{d_0}. \end{aligned} \tag{6.1}$$

This definition is best understood by looking at the portraits on Figure 3. The automorphism subgroup generated is denoted $H_{\bar{d}} = \langle x_{\bar{d}}, y_{\bar{d}}\rangle$. Note that in the case $\bar{d} = \sigma\bar{d}$ is a constant sequence the group $H_{\bar{d}}$ is generated by a two (non trivial) state automaton.

Property 6.2. *The alternate automorphism $x_{\bar{d}}$ has order 2, and $y_{\bar{d}}$ has order 3.*

Proof. Show by joint (on $x_{\sigma^i\bar{d}}$ for i in $\mathbb{N}$) induction on k that $x_{\sigma^i\bar{d}}^2$ acts trivially on the k first levels of $T_{\sigma^i\bar{d}}$. This implies it acts trivially on the whole tree hence is trivial automorphism. Proposition 6.1 1) ensures:

$$x_{\sigma^i\bar{d}}^2 = (x_{\sigma^i\bar{d}}^2, 1, \dots, 1)x_{d_i}^2 = (x_{\sigma^i\bar{d}}^2, 1, \dots, 1),$$

which initiates the induction. Moreover $x_{\sigma^i\bar{d}}^2$ acts trivially on the subtrees $T_2, \dots, T_{d_i}$ of $T_{\sigma^i\bar{d}}$ and as $x_{\sigma^{i+1}\bar{d}}^2$ on T_1 which acts trivially on the k first level of T_1 by induction hypothesis. This proves $x_{\sigma^i\bar{d}}^2$ acts trivially on the $k+1$ first levels of $T_{\sigma^i\bar{d}}$. $\square$

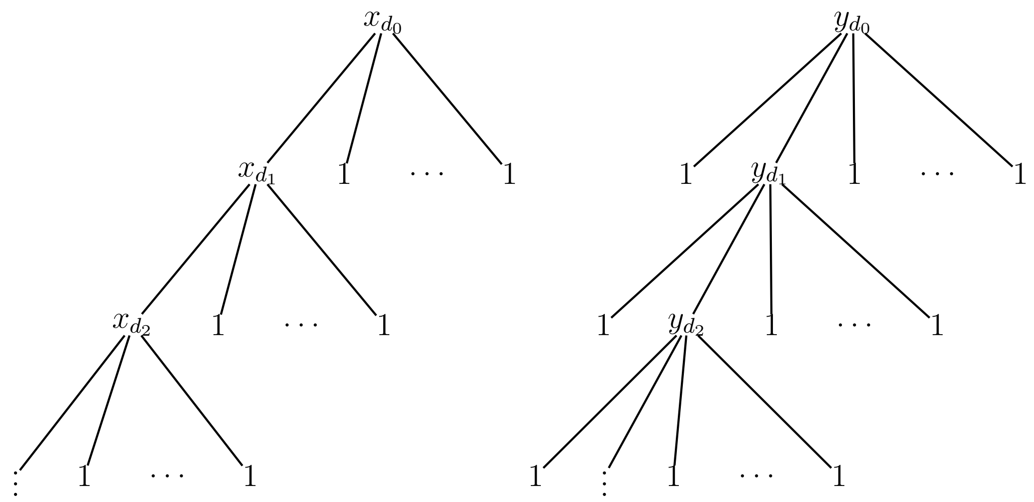

FIGURE 3. Portraits of the elements $x_{\bar{d}}$ and $y_{\bar{d}}$.

6.2. **Density properties.**

Proposition 6.3. *The subgroup $H_{\bar{d}} = \langle x_{\bar{d}}, y_{\bar{d}}\rangle < Aut^e(T_{\bar{d}})$ is dense in $Aut^e(T_{\bar{d}})$ endowed with the profinite topology from (2.4).*

Proof. It is sufficient to show that the subgroup $\mathcal{A}_{d_k} \wr \cdots \wr \mathcal{A}_{d_0} < Aut^e(T_{\bar{d}})$ of alternate automorphisms of portrait supported on the k firsts levels is included in $H_{\bar{d}}$ for arbitrary k. Proceed by joint (on $H_{\sigma^i\bar{d}}$ for $i \in \mathbb{N}$) induction on k to show:

$$H_{\sigma^i\bar{d}} \simeq H_{\sigma^{i+k}\bar{d}} \wr \mathcal{A}_{d_{i-1}} \wr \cdots \wr \mathcal{A}_{d_i}, \tag{6.2}$$

which will be sufficient taking $i = 0$ and the trivial subgroup of $H_{\sigma^k\bar{d}}$. The case $k = 0$ follows from Proposition 6.1 2):

$$H_{\sigma^i\bar{d}} = \langle x_{\sigma^i\bar{d}}, y_{\sigma^i\bar{d}}\rangle \simeq \langle x_{\sigma^{i+1}\bar{d}}, y_{\sigma^{i+1}\bar{d}}\rangle \wr \mathcal{A}_{d_i} = H_{\sigma^{i+1}\bar{d}} \wr \mathcal{A}_{d_i}. \tag{6.3}$$

Assuming isomorphism (6.2) then isomorphism (6.3) for $i+k$ proves step $k+1$:

$$H_{\sigma^i\bar{d}} \simeq H_{\sigma^{i+k}\bar{d}} \wr \mathcal{A}_{d_{i+k-1}} \wr \cdots \wr \mathcal{A}_{d_i} \simeq H_{\sigma^{i+k+1}\bar{d}} \wr \mathcal{A}_{d_{i+k}} \wr \mathcal{A}_{d_{i+k-1}} \wr \cdots \wr \mathcal{A}_{d_i}.$$

□

This density property is in contrast with the case of the full (non alternate) automorphism group of a rooted tree:

Proposition 6.4. *The group $Aut(T_{\bar{d}})$ endowed with the profinite topology from (2.3) admits no finitely generated dense subgroup.*

Proof. Denote $sgn : S_d \to \mathbb{Z}/2\mathbb{Z}$ the signature morphism of permutations. Given an element g in $Aut(T_{\bar{d}})$, recall that $g(v)$ is the permutation in $S_{d_{l(v)}}$ associated to vertex v in the portrait of g.

(Recall $g = (g_{1\ldots1}, \ldots, g_v, \ldots, g_{d_0\ldots d_{l(v)-1}})\tau_{l(v)-1}$ with g_v in $Aut(T_v) \simeq Aut(T_{\sigma^{l(v)}\bar{d}})$ and $\tau_{l(v)-1} \in Aut(T_{d_0\ldots d_{l(v)-1}})$, then g_v has image $g_v = (g_{v1}, \ldots, g_{vd_{l(v)}})g(v)$ via the isomorphism $Aut(T_{\sigma^{l(v)}\bar{d}}) \simeq Aut(T_{\sigma^{l(v)+1}\bar{d}}) \wr S_{d_{l(v)}}$.)

Similarly to Lemma 1. in [Ale2], define for each integer k the following morphism (of products of signatures of permutations on level k in the portraits):

$$\begin{array}{rccl} R_k: & Aut(T_{\bar{d}}) & \to & \mathbb{Z}/2\mathbb{Z} \\ & g & \mapsto & R_k(g) = \prod_{v \in Level(k)} sgn(g(v)). \end{array}$$

The computations via the isomorphism (2.2) show this is a group morphism. The product morphism $\varphi : Aut(T_{\bar{d}}) \to (\mathbb{Z}/2\mathbb{Z})^\infty$ defined as $\varphi(g) = (R_0(g), R_1(g), \dots)$ is then a surjective group morphism continuous for the profinite topologies. Assume now there exists a finitely generated dense subgroup G of $Aut(T_{\bar{d}})$, then $\varphi(G)$ is a finitely generated dense subgroup of $(\mathbb{Z}/2\mathbb{Z})^\infty$. This is impossible since any finitely generated subgroup of $(\mathbb{Z}/2\mathbb{Z})^\infty$ is finite thanks to Fact 3.4. $\square$

Density in $Aut^e(T_{\bar{d}})$ of a finitely generated subgroup implies superpolynomial growth:

Proposition 6.5. *Let $\bar{d} = (d_i)_{i \in \mathbb{N}}$ a sequence of integers $d_i \geq 3$, then any dense finitely generated subgroup of $Aut^e(T_{\bar{d}})$ has superpolynomial growth.*

Proof. Let G be such a group and k an arbitrary integer, then the level k stabilizer $St_k(G) \simeq G_{1\dots1} \times \cdots \times G_{d_0 \dots d_{k-1}}$ is a direct product of $d_0 \dots d_{k-1}$ subgroups of $Aut^e(T_{\sigma^k \bar{d}})$ each of which inherits the property to be dense and finitely generated. In particular each of the groups G_v is infinite ($d_i \geq 3$) and thus has at least linear growth, so that the subgroup $St_k(G)$ of finite index and thus G have growth function at least $b(r) \gtrsim r^{d_0 \dots d_{k-1}}$, hence superpolynomial. $\square$

6.3. **Intermediate growth.**

Proposition 6.6. *The group $H_{\bar{d}} < Aut(T_{\bar{d}})$ has intermediate growth.*

Proof of Proposition 6.6. Superpolynomial growth follows from Propositions 6.3 and 6.5, so there remains to prove subexponential growth. Proceed as in [Gri1]. Denote $B_k(r)$ the ball of radius r in $H_{\sigma^k \bar{d}}$ for the word metric $|.|_k$ associated with the generating set $\langle x_{\sigma^k \bar{d}}, y_{\sigma^k \bar{d}} \rangle$, denote $b_k(r)$ its cardinal and $c_k = \lim \sqrt[r]{b_k(r)} = h_{\{x_{\sigma^k \bar{d}}, y_{\sigma^k \bar{d}}\}}(H_{\sigma^k \bar{d}})$ its exponential growth rate. The fixed point condition on eligible pairs ensures:

$$x_{\bar{d}} y_{\bar{d}} x_{\bar{d}} y_{\bar{d}}^{-1} x_{\bar{d}} = (x_{\sigma \bar{d}}, 1, \dots, y_{\sigma \bar{d}}, y_{\sigma \bar{d}}^{-1}, \dots, 1) x_{d_0} y_{d_0} x_{d_0} y_{d_0}^{-1} x_{d_0}, \tag{6.4}$$

with $y_{\sigma \bar{d}}$ in positions $x_{d_0}(2)$ and $x_{d_0} y_{d_0} x_{d_0}(2)$ and the second and third $x_{\sigma \bar{d}}$ cancel out. As the generators are of order 2 and 3 every element $g = (g_1, \dots, g_{d_0})\sigma$ in $B_0(r)$ admits a minimal representative word of the form $g = x_{\bar{d}} y_{\bar{d}}^{\varepsilon_1} x_{\bar{d}} y_{\bar{d}}^{\varepsilon_2} \dots x_{\bar{d}} y_{\bar{d}}^{\varepsilon_n} x_{\bar{d}}$, with ε_i in $\{-1, 1\}$. Given g (more precisely given a fixed minimal representative word), denote $a(g)$ the number of alternations in the sequence (ε_i), equality (6.4) implies:

$$|g_1|_1 + \cdots + |g_{d_0}|_1 \leq |g|_0 - a(g). \tag{6.5}$$

Given any parameter $t \geq 2$, split the ball $B_0(r)$ into:

$$\begin{array}{l} B_0^+(r) = \{g \in B_0(r) | a(g) \geq \frac{r}{t}\}, \\ B_0^-(r) = \{g \in B_0(r) | a(g) \leq \frac{r}{t}\}. \end{array}$$

The size of the first part of the ball is bounded by:

$$b_0^+(r) \leq \#\mathcal{A}_{d_0} \sum_{r_1+\dots+r_{d_0}\leq(1-\frac{1}{t})r} b_1(r_1)\dots b_1(r_{d_0}). \tag{6.6}$$

Indeed, each element $g = (g_1, \dots, g_{d_0})\sigma$ of $B_0(r)$ is injectively described by the permutation σ in $\mathcal{A}_{d_0}$ and the coordinates $g_1, \dots, g_{d_0}$ the sum of the $|.|_1$ length is bounded by $r - a(g) \leq (1-\frac{1}{t})r$ thanks to computation (6.5). The size of the second part of the ball is bounded by (recall notation C_n^k for the number of subsets of size k in $\{1, \dots, n\}$):

$$b_0^-(r) \leq 4 \sum_{s\leq\frac{r}{t}} C_r^s \leq 4\frac{r}{t}C_r^{\frac{r}{t}}. \tag{6.7}$$

Indeed the term 4 corresponds to choosing the start of the representative word (y, y^{-1}, xy or xy^{-1}), s represents the number of alternation $a(g)$ and C_r^s the number of choice for the positions of such alternations.

The size is estimated by $b_0(r) \leq b_0^+(r)+b_0^-(r) \leq \max\{2b_0^+(r), 2b_0^-(r)\}$, and taking limits of r-roots raises $c_0 \leq \max\{c_1^{1-\frac{1}{t}}, t^{\frac{1}{t}}(1-\frac{1}{t})^{(\frac{1}{t}-1)}\}$, since Stirling formula ensures:

$$(4\frac{r}{t}C_r^{\frac{r}{t}})^{\frac{1}{r}} \sim_{r\to\infty} \left(4\frac{r}{t}\frac{\sqrt{(2\pi r)}}{\sqrt{2\pi\frac{r}{t}}\sqrt{2\pi(1-\frac{1}{t})r}}\right)^{\frac{1}{r}} \frac{\frac{r}{e}}{(\frac{r}{et})^{\frac{1}{t}}((1-\frac{1}{t})\frac{r}{e})^{1-\frac{1}{t}}} \sim_{r\to\infty} t^{\frac{1}{t}}(1-\frac{1}{t})^{\frac{1}{t}-1}.$$

The estimate is valid for any level k so that for all parameter $t \geq 2$:

$$c_k \leq \max\{c_{k+1}^{1-\frac{1}{t}}, t^{\frac{1}{t}}(1-\frac{1}{t})^{\frac{1}{t}-1}\}.$$

In particular, this shows the sequence $(c_k)_k$ increases (note $t^{\frac{1}{t}}(1-\frac{1}{t})^{\frac{1}{t}-1} \to 1$ for $t \to \infty$). Moreover the sequence is bounded by 2 (the groups are quotients of $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$), hence admits a limit c_∞, which satisfies by continuity:

$$c_\infty \leq \max\{c_\infty^{1-\frac{1}{t}}, t^{\frac{1}{t}}(1-\frac{1}{t})^{\frac{1}{t}-1}\}$$

for any parameter $t \geq 2$, which is impossible unless $c_\infty = 1$ (otherwise take t large enough). This shows subexponential growth of the groups $H_{\sigma^k\bar{d}}$. □

Remark 6.7. When the tree has bounded valency, set $f(r) = \max\{b_k(r) | k \in \mathbb{N}\}$ the estimate (6.6) can be made homogeneous on $d_k \leq D$. This together with estimate (6.7) applied for a parameter t of the form $t = \frac{K}{\log(r)}$ raises inequality:

$$f(r) \leq K\left(\sum_{r_1+\dots+r_D\leq(1-\frac{K}{\log(r)})r} \prod_{i=1}^{D} f(r_i)\right) + KC_r^{\frac{K}{\log(r)}}.$$

A computation due to Erschler (Lemma 6.4 in [Ers]) gives the explicit upper bound on the growth:

$$b_0(r) \leq f(r) \leq \exp\left(\frac{K\log(\log(r))r}{\log(r)}\right).$$

7. Groups of non uniform growth

7.1. **A Theorem of Wilson.** The first examples of groups of non uniform exponential growth have been constructed by Wilson in [Wil1]. The following Theorem from [Wil2] is a generalization.

Theorem 7.1 (Wilson [Wil2]). *Let k be a positive integer and χ_k a class of groups with the two properties:*

(1) each group G in χ_k is perfect (that is $G = [G, G]$) and can be generated by k involutions;
(2) each group G in χ_k is isomorphic to a permutational wreath product $G_1 \wr \mathcal{A}_d$ with $G_1 \in \chi_k$ and $d \geq 29$.

Then each group G in χ_k contains two sequences of elements $(a^{(n)})$, $(b^{(n)})$ such that:

(a) $(a^{(n)})^2 = (b^{(n)})^3 = 1$ and $\langle a^{(n)}, b^{(n)}\rangle = G$ for each n and,
(b) $h_{\{a^{(n)},b^{(n)}\}}(G) \to 1$ as $n \to \infty$.

In section 4. of [Wil2], Wilson constructs subgroups of $Aut^e(T_{\bar{d}})$ for unbounded sequences $\bar{d} = (d_i)_i$ in the classes χ_k. Unboundedness of the sequence permits to construct such groups with a subgroup isomorphic to the free group $\mathbb{F}_2$ on two generators. This ensures exponential growth, but prevents amenability.

In the next section groups in the class χ_k are constructed similarly but acting on bounded valency rooted tree. The Main Theorem 3.1 will apply to show amenability. Exponential growth is due to the presence of free semigroups. Note however that in [Wil1] Wilson constructs groups of automorphism of a regular (in particular bounded valency) rooted tree which have non uniform growth and contain a free group.

7.2. **Amenable groups of non uniform growth.** Let $\bar{d} = (d_i)_i$ be a bounded sequence of integers $5 \leq d_i \leq D$, define a subgroup of the group $\bar{H}$ (constructed in section 2.3) as $\bar{A} < \bar{H} = S_{d_1}^{d_0-1} \times S_{d_2}^{d_1-1} \times \dots$ where $\bar{A} = \mathcal{A}_{d_1} \times \mathcal{A}_{d_2} \times \dots$ as an abstract group and each group $\mathcal{A}_{d_k}$ is acting as a rooted automorphism on $T_{1_{k-1}2}$; this is best understood by Figure 4.

Now for each integer d in $\{5, \dots, D\}$, denote $E_d = \{i \geq 1 | d_i = d\}$. There is a diagonal injection:

$$j_d : \mathcal{A}_d \hookrightarrow \prod_{i \in E_d} \mathcal{A}_{d_i} < \bar{A},$$

and the diagonal product of those injections:

$$j : A_{\bar{d}} = \prod_{d=5}^{D} \mathcal{A}_d \hookrightarrow \bar{A}.$$

To ease notation the image subgroup of $A_{\bar{d}}$ is still denoted $A_{\bar{d}}$. It is a finite saturated subgroup of $\bar{A}$. The subgroup of $Aut^e(T_{\bar{d}})$ generated by alternate rooted automorphisms $\mathcal{A}_{d_0}$ and $A_{\bar{d}}$ is denoted $G_0 = G(\mathcal{A}_{d_0}, A_{\bar{d}}) < Aut^e(T_{\bar{d}})$. Note that when $\bar{d} = \sigma\bar{d}$ is a constant sequence, the group $G(\mathcal{A}_d, A_{\bar{d}})$ is generated by a finite automaton.

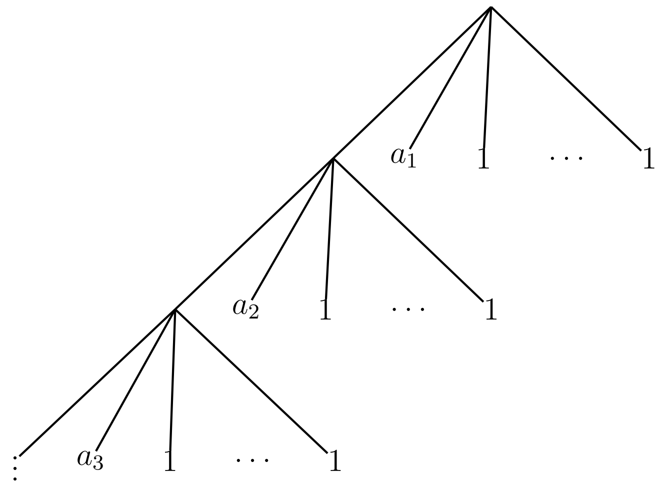

FIGURE 4. The group $\bar{A}$.

Proposition 7.2. *Let $\bar{d} = (d_i)_i$ a bounded sequence of integers $29 \leq d_i \leq D$, the group $G_0 = G(\mathcal{A}_{d_0}, A_{\bar{d}})$ belongs to the class χ_k where k depends only on D.*

Proof. Show this Proposition simultaneously for all groups $G_i = G(\mathcal{A}_{d_i}, A_{\sigma^i \bar{d}}) < Aut^e(T_{\sigma^i \bar{d}})$. The group G_i is perfect because generated by copies of the groups $\mathcal{A}_{d_i}$, $\mathcal{A}_d, d \in \{5, \dots, D\}$ which are perfect (even simple). Moreover, those groups (hence G_i) are generated by double transpositions, in particular by involutions the number of which depends only on D, so that the condition (1) of definition of groups in the class χ_k is satisfied for some k depending only on D.

To check condition (2), note first that the injection in the wreath product (2.1) has image in:

$$G_i = G(\mathcal{A}_{d_i}, A_{\sigma^i \bar{d}}) \hookrightarrow G(\mathcal{A}_{d_{i+1}}, A_{\sigma^{i+1} \bar{d}}) \wr \mathcal{A}_{d_i} = G_{i+1} \wr \mathcal{A}_{d_i}. \tag{7.1}$$

This is clear for the generators in $\mathcal{A}_{d_i}$ and the generators b in $A_{\sigma^i \bar{d}}$ have image $b = (b', a, 1, \dots, 1)$ where a belongs to $\mathcal{A}_{d_{i+1}}$ and b' to $A_{\sigma^{i+1} \bar{d}}$ by construction. Now remains to prove this injection is onto hence an isomorphism.

Given any two elements a_1, a_2 in $\mathcal{A}_{d_{i+1}}$ there exists $b_1 = (b'_1, a_1, 1, \dots, 1)$ and $b_2 = (b'_2, a_2, 1, \dots, 1)$ in $A_{\sigma^i \bar{d}}$. Moreover the double transposition $\sigma = (13)(45)$ belongs to $\mathcal{A}_{d_i}$, so that G_i contains $b_2^\sigma = \sigma b_2 \sigma^{-1} = (1, a_2, b'_2, 1, \dots, 1)$, hence $[b_1, b_2^\sigma] = (1, [a_1, a_2], 1, \dots, 1)$ and then $1 \times \mathcal{A}_{d_{i+1}} \times \dots \times 1$ by perfection. Similarly given any two b'_1, b'_2 in $A_{\sigma^{i+1} \bar{d}}$, the group G_i contains $[b_1, b_2^\tau] = ([b'_1, b'_2], 1, \dots, 1)$ where $\tau = (23)(45)$, hence $A_{\sigma^{i+1} \bar{d}} \times 1 \times \dots \times 1$. Since $\mathcal{A}_{d_i}$ acts transitively by conjugation on the coordinates, this proves injection (7.1) is onto. □

Proposition 7.3. *Let $\bar{d} = (d_i)_i$ a bounded sequence of integers $5 \leq d_i \leq D$, the group $G_0 = G(\mathcal{A}_{d_0}, A_{\bar{d}})$ has exponential growth.*

Proof. Each group $\mathcal{A}_d$ contains the double transpositions $u = (12)(34)$ and $v = (12)(35)$. Moreover each of the groups $A_{\sigma^i \bar{d}} \simeq \mathcal{A}_{d_i} \times \mathcal{A}_{d_{i+1}} \times \dots$ contains the diagonal elements $\bar{u} = (u, u, u, \dots)$ and $\bar{v} = (v, v, v, \dots)$. The following Lemma due to

Bartholdi (Proposition 2.3 in [Bar]) ensures that $\langle \bar{u}u, \bar{v}v\rangle \simeq \mathbb{S}_2$ is a free semigroup. More precisely:

Lemma 7.4 (Bartholdi [Bar])**.** *The quotient semigroup*

$$\langle \bar{u}u, \bar{u}v, \bar{v}u, \bar{v}v\rangle/(\bar{u}u = \bar{u}v, \bar{v}u = \bar{v}v) \simeq \mathbb{S}_2$$

is freely generated by $\{\bar{u}u, \bar{v}v\}$.

This ensures exponential growth of the group G_0. □

Corollary 7.5 (Theorem 1.1)**.** *The groups* $G(\mathcal{A}_{d_0}, A_{\bar{d}})$ *associated to sequences* $\bar{d} = (d_i)_i$ *of integers* $29 \leq d_i \leq D$ *are (uncountably many pairwise non isomorphic) amenable groups of non uniform exponential growth.*

Proof. This follows from the Main Theorem 3.1, Wilson's Theorem 7.1, Proposition 7.2 and Proposition 7.3. The bracketted part follows from Corollary 8.2. □

7.3. **Convergence of the Cayley graphs.** This section is devoted to give another proof of some part of Wilson Theorem 7.1. Namely the convergence to 1 of the exponential growth rate of the generating sets $\{a^{(n)}, b^{(n)}\}$ can be understood as the convergence of the associated Cayley graphs of the group to the Cayley graph of a group $H_{\bar{d}}$ of intermediate growth introduced in section 6.

More precisely, let $G = G_0$ belong to some class χ_k, then by definition of the class there exists a sequence of groups G_i in χ_k and integers $d_i \geq 29$ such that $G_i \simeq G_{i+1} \wr \mathcal{A}_{d_i}$. The Theorem 7.1 of Wilson ensures in particular that for each integer i there exists a generating pair of elements $\langle a_i^{(0)}, b_i^{(0)}\rangle = G_i$ such that $(a_i^{(0)})^2 = (b_i^{(0)})^3 = 1$. Out of this first generating pair, Wilson constructs a sequence of generating pairs for G_i, defined inductively as (also see Figure 5 and compare with Figure 3):

$$\begin{aligned} a_i^{(n+1)} &= (a_{i+1}^{(n)}, 1, \dots, 1)x_{d_i}, \\ b_i^{(n+1)} &= (1, b_{i+1}^{(n+1)}, 1, \dots, 1)y_{d_i}. \end{aligned} \tag{7.2}$$

The fact that $a_i^{(n)}$ and $b_i^{(n)}$ have order 2 and 3 and that they generate G_i is a direct consequence of the properties of the generating pairs x_{d_i}, y_{d_i} of the alternate group $\mathcal{A}_{d_i}$ (see Proposition 6.1).

Definition 7.6 (Distance between Cayley graphs)**.** Let (Γ, S) and (Δ, T) be two groups with generating sets, denote $B_{\Gamma,S}(R)$ the restriction of the Cayley graph of Γ relatively to the generating set S to vertices at distance less than R of the neutral element (for the word distance in S). The distance between (Γ, S) and (Δ, T) is defined as:

$$d((\Gamma, S), (\Delta, T)) = \inf\{\frac{1}{R} | B_{\Gamma,S}(R) \sim_G B_{\Delta,T}(R)\},$$

where $Gr_1 \sim_G Gr_2$ if Gr_1 and Gr_2 are isometric as colored graphs.

Non uniform growth of the group G_0 comes from the two next propositions, since intermediate growth of $H_{\bar{d}}$ implies $h_{\{x_{\bar{d}}, y_{\bar{d}}\}}(H_{\bar{d}}) = 1$.

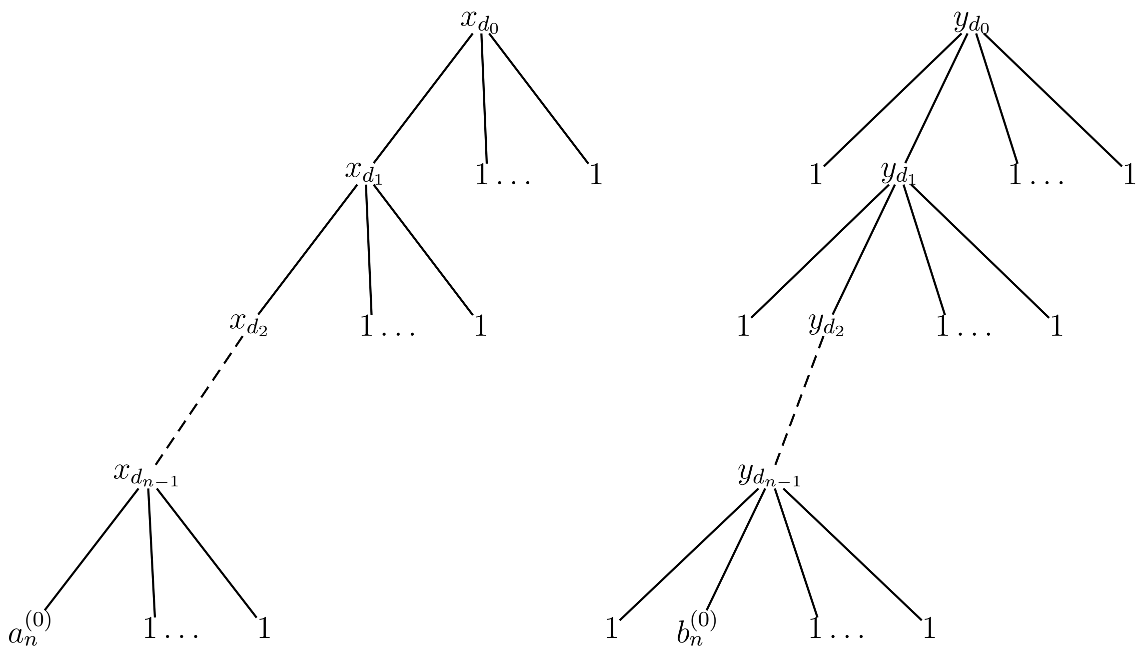

FIGURE 5. Portraits of the elements $a_0^{(n)}$ and $b_0^{(n)}$.

Proposition 7.7. *With the notations above:*

$$d((G_0, \{a_0^{(n)}, b_0^{(n)}\}), (H_{\bar{d}}, \{x_{\bar{d}}, y_{\bar{d}}\})) \underset{n\to+\infty}{\longrightarrow} 0.$$

Note that this Proposition is true independently of the amenability or not of the group G_0 in a class χ_k. In particular, such a convergence is also true for the non amenable groups constructed by Wilson in [Wil1], [Wil2].

Proposition 7.8. *If* $d((\Gamma, S_n), (\Delta, T)) \to 0$*, then:*

$$\limsup_{n\to\infty} h_{S_n}(\Gamma) \leq h_T(\Delta).$$

Proof of Proposition 7.8. Given a positive ε the definition of $h_T(\Delta)$ ensures that for $R \geq R_0$ large enough the ball $B_{\Delta,T}(R)$ has size $\#B_{\Delta,T}(R) \leq (h_T(\Delta)+\varepsilon)^R$. Now the convergence of Cayley graphs shows that for $n \geq N$ large enough $\#B_{\Gamma,S_n}(R) \leq (h_T(\Delta)+\varepsilon)^R$, and by subadditivity $\#B_{\Gamma,S_n}(kR) \leq \#B_{\Gamma,S_n}(R)^k \leq (h_T(\Delta)+\varepsilon)^{kR}$, so that:

$$h_{S_n}(\Gamma) = \lim \sqrt[kR]{\#B_{\Gamma,S_n}(kR)} \leq h_T(\Delta) + \varepsilon,$$

which was required. $\square$

The proof of Proposition 7.7 uses the following:

Lemma 7.9 (of contraction)**.** *If* $\hat{x} = (u, 1, \ldots, 1)x_d$ *and* $\hat{y} = (1, v, 1, \ldots, 1)y_d$ *are as in Proposition 6.1, then for elements* $g = (g_1, \ldots, g_d)\sigma$ *in the wreath product isomorphism* $\langle \hat{x}, \hat{y}\rangle \simeq \langle u, v\rangle \wr \mathcal{A}_d$*, one has for each coordinate* t*:*

$$|g_t|_{\{u,v\}} \leq \frac{1}{2}(|g|_{\{\hat{x},\hat{y}\}} + 1),$$

where $|.|_S$ denotes the word norm associated to the generating set S (inverses of elements of S have length 1).

Proof. It is sufficient to check that $\hat{x}\hat{y}^\varepsilon = (u, 1, \ldots, v^\varepsilon, 1, \ldots, 1)x_d y_d$ with v^ε on coordinate $x_d(2) \neq 1$. $\square$

Proof of Proposition 7.7. Introduce other relations depending on integer $l \geq 1$ on groups with generating sets: $(\Gamma, S) \sim_l (\Delta, T)$ if for every free word w of length less than l in S (elements and inverses) one has $w(S) = id_\Gamma$ if and only if $w(T) = id_\Delta$ (for $l = 1$ the relation $\sim_1$ just means $S \cup S^{-1}$ and $T \cup T^{-1}$ have the same size). If the relation $(\Gamma, S) \sim_{2l+1} (\Delta, T)$ is satisfied then $d((\Gamma, S), (\Delta, T)) \leq \frac{1}{l}$ because to describe $B_{\Gamma,S}(R)$ it is sufficient to know when $g'g^{-1} = s$ for every g, g' in $B_{\Gamma,S}(R)$ and s in $S \cup S^{-1}$.

To ease notations set $S_i^{(n)} = \{a_i^{(n)}, b_i^{(n)}\}$ and $T_i = \{x_{\sigma^i \bar{d}}, y_{\sigma^i \bar{d}}\}$. It is sufficient to show for all integers i: $(G_i, S_i^{(n)}) \sim_{l_n} (H_{\sigma^i \bar{d}}, T_i)$ with a sequence $l_n \to \infty$. Proceed by induction on n, using:

$$\begin{aligned} w(S_i^{(n+1)}) &= (w_1(S_{i+1}^{(n)}), \ldots, w_{d_i}(S_{i+1}^{(n)}))w(x_{d_i}, y_{d_i}), \\ w(T_i) &= (w_1(T_{i+1}), \ldots, w_{d_i}(T_{i+1}))w(x_{d_i}, y_{d_i}), \end{aligned}$$

where for each coordinate t the elements $w_t(S_{i+1}^{(n)})$ and $w_t(T_{i+1})$ involve the same word w_t because the permutations on the first level are the same for generators in $S_i^{(n+1)}$ or in T_i (namely x_{d_i} and y_{d_i}). Lemma 7.9 ensures that $|w_t(T_{i+1})| \leq \frac{1}{2}(|w(T_i)| + 1)$ (and $|w_t(S_{i+1}^{(n)})| \leq \frac{1}{2}(|w(S_i^{(n+1)})| + 1)$) so that if w has length less than $l_{n+1} = 2l_n - 1$ one has $w(S_i^{(n+1)}) = id_{G_i}$ if and only if $w(T_i) = id_{H_{\sigma^i \bar{d}}}$. The result follows since the sequence (l_n) starts with $l_0 = 1$ and $l_1 = 2$. $\square$

Corollary 7.10 (of Proposition 7.7)**.** *The group $H_{\bar{d}}$ of intermediate growth is not finitely presented.*

Proof. Assume the contrary $H_{\bar{d}} = \langle x_{\bar{d}}, y_{\bar{d}} | r_1, \ldots, r_k \rangle$. Let R be bigger than the maximal length of the relations $r_1, \ldots, r_k$, and n large enough so that:

$$d((G_0, \{a_0^{(n)}, b_0^{(n)}\}), (H_{\bar{d}}, \{x_{\bar{d}}, y_{\bar{d}}\})) \leq \frac{1}{R}.$$

Then the automorphisms $a_0^{(n)}$ and $b_0^{(n)}$ satisfy all relation $r_1, \ldots, r_k$. In particular, G_0 is a quotient of $H_{\bar{d}}$ hence has subexponential growth. This contradicts Proposition 7.3. $\square$

8. Non subexponential amenability

8.1. **Description of normal subgroups.** The normal subgroups of finite index of groups in a class χ_k are completely described by the:

Proposition 8.1 (Neumann [Neu])**.** *Let $(G_i)_{i \in \mathbb{N}}$ be a sequence of finitely generated perfect groups such that for each i there exists an integer $d_i \geq 5$ such that $G_i \simeq$*

$G_{i+1} \wr \mathcal{A}_{d_i}$. *Consider the isomorphisms:*

$$G_0 \simeq G_i \wr \mathcal{A}_{d_{i-1}} \wr \cdots \wr \mathcal{A}_{d_0} \simeq \underbrace{(G_i \times \cdots \times G_i)}_{d_0 \ldots d_{i-1} times} \rtimes Aut^e(T_{d_0 \ldots d_{i-1}}),$$

then the subgroups $K_i = (G_i \times \cdots \times G_i)$ for $i \in \mathbb{N}$ are the only normal subgroups of G_0 of finite index. Moreover if one (hence all) of the groups G_i is residually finite, then $(K_i)_{i \in \mathbb{N}}$ are the only non trivial normal subgroups of G_0; in particular G_0 is just infinite.

Proposition 8.1 (as well as Lemma 8.3) is a slight generalization of Theorem 5.1. in [Neu]. The proof is given here for the sake of completeness and to avoid the reader multiple references and notations. The second part is also similar to Theorem 4. in [Gri2]. Note that all examples in this paper are groups of automorphism of a rooted tree. In particular they satisfy the assumption of residual finiteness.

Corollary 8.2. *Two groups G_0 and H_0 satisfying the hypothesis of Proposition 8.1 (in particular groups in a class χ_k) for two different sequences of integers $(d_i)_i$ and $(e_i)_i$ are non isomorphic.*

Proof. The index of K_i in G_0 has value:

$[G_0 : K_i] = \#Aut^e(T_{d_0 \ldots d_{i-1}}) = \#(\mathcal{A}_{d_{i-1}} \wr \cdots \wr \mathcal{A}_{d_0}) = a(d_{i-1})^{d_{i-2} \ldots d_0} \ldots a(d_1)^{d_0} a(d_0),$

where $a(d) = \frac{d!}{2} = \#\mathcal{A}_d$. In particular the sequence of index of subgroups $([G_0 : K_i])_i$ is an isomorphism invariant from which the sequence $(d_i)_i$ can be recovered. □

Lemma 8.3. *Under the hypothesis of Proposition 8.1, the only normal subgroups of G_0 containing K_m are $K_0, K_1, \ldots, K_m$.*

The proof of this lemma will use the:

Fact 8.4. *Given a finite group Γ, assume $\Delta \lhd \Gamma$ is a minimal normal subgroup (minimal means the only normal subgroup of Γ strictly contained in Δ is trivial) and that the centralizer $Cent_\Gamma(\Delta)$ of Δ is trivial. Then Δ is the unique minimal normal subgroup of Γ.*

Proof. Assume Δ' is another such subgroup, then $\Delta \cap \Delta'$ is trivial by minimality. In particular, for every $\delta \in \Delta$ and $\delta' \in \Delta'$ the commutator $\Delta \cap \Delta' \ni [\delta, \delta'] = 1$, which ensures $\Delta' \subset Cent_\Gamma(\Delta) = \{1\}$, contradiction. □

Proof of Lemma 8.3. By induction on m and using Fact 8.4, it is sufficient to prove that:

$$\mathcal{A}_{d_{m-1}}^{(1 \ldots 1)} \times \cdots \times \mathcal{A}_{d_{m-1}}^{(d_0 \ldots d_{m-2})} \simeq K_{m-1}/K_m \lhd G_0/K_m \simeq Aut^e(T_{d_0 \ldots d_{m-1}})$$

is minimal and has trivial centralizer, which shows K_{m-1} is the only minimal subgroup of G_0 containing K_m.

Let U a non trivial subgroup normal in G_0/K_m and included in K_{m-1}/K_m. Then $1 \neq y \in U$ can be written $y = (y_{1 \ldots 1}, \ldots, y_{d_0 \ldots d_{m-1}})$ in the wreath product $G_0/K_m \simeq \mathcal{A}_{d_{m-1}} \wr Aut^e(T_{d_0 \ldots d_{m-2}})$, with some coordinate $1 \neq y_v \in \mathcal{A}_{d_{m-1}}^{(v)}$. By simplicity, the normal closure of y_v is the full alternate group $\langle y_v \rangle_{\mathcal{A}_{d_{m-1}}} = \mathcal{A}_{d_{m-1}}$. Moreover the group $Aut^e(T_{d_0\dots d_{m-2}})$ acts by conjugation transitively on the coordinates, so that $U > \langle y\rangle_{G_0/K_m} = \mathcal{A}_{d_{m-1}}^{(1\dots 1)} \times \cdots \times \mathcal{A}_{d_{m-1}}^{(d_0\dots d_{m-2})} = K_{m-1}/K_m$, proving minimality. Transitivity also shows that the centralizer $Cent_{G_0/K_m}(K_{m-1}/K_m)$ is included in $St_{m-1}(G_0/K_m) = K_{m-1}/K_m$, which has trivial center, hence the centralizer is trivial. □

Proof of Proposition 8.1. Let X be a finite group and $f : G_0 \to X$ a homomorphism. Restricting to factors of the subgroups $K_m = G_m^{(1\dots 1)} \times \cdots \times G_m^{(v)} \times \cdots \times G_m^{(d_0\dots d_{m-1})}$, it appears that for m large enough there exists $v \neq v'$ such that the associated factors have the same image $f(G_m^{(v)}) = f(G_m^{(v')}) = Y$, which must be abelian because $[G_m^{(v)}, G_m^{(v')}] = 1$, hence $Y = \{1\}$ because $G_m^{(v)} \simeq G_m$ is perfect. This shows $G_m^{(v)} \subset \mathrm{Ker}(f)$.

Moreover for each coordinate v' there exists $\varphi \in Aut^e(T_{d_0\dots d_{m-1}})$ such that $\varphi(v) = v'$, so that $\varphi G_m^{(v)} \varphi^{-1} = G_m^{(v')} \subset \mathrm{Ker}(f)$ and consequently K_m lies in the kernel of f. Applying Lemma 8.3 shows $\mathrm{Ker}(f) = K_i$ for some $i \leq m$, which proves the first part.

Now assume G_0 is residually finite, and $N \lhd G_0$ is an arbitrary normal subgroup. The description of the first part ensures that $\cap_{m\geq 0} K_m = \{1\}$, and as the sequence of subgroups $(K_m)_m$ is strictly decreasing there exists an integer n such that $N \leq K_n$ and $N \nleq K_{n+1}$. To get the second part, it is sufficient to prove $N \geq K_{n+1}$ since the first part will force $K_n = N$.

Consider $x \in N \setminus K_{n+1}$ and its image $x = (x_{1\dots 1}, \dots x_{d_0\dots d_{n-1}})_n$ in the factor decomposition of K_n. There exists v such that:

$$x_v = (x_{v1}, \dots, x_{vd_n})\sigma_v \in G_n^{(v)} \simeq G_{n+1} \wr \mathcal{A}_{d_n},$$

with a non trivial permutation σ_v, and in particular there are $s \neq t$ in $\{1, \dots d_n\}$ with $\sigma_v(s) = t$. Now given any two elements ξ, η in G_{n+1}, define f, g in $K_n = (G_n \times \cdots \times G_n)$ as:

$$\begin{array}{ll} f = (1, \dots, 1, f_v, 1 \dots, 1)_n, & f_v = (1, \dots, 1, \xi, 1, \dots, 1) \in G_{n+1} \wr \mathcal{A}_{d_n}, \\ g = (1, \dots, 1, g_v, 1 \dots, 1)_n, & g_v = (1, \dots, 1, \eta, 1, \dots, 1) \in G_{n+1} \wr \mathcal{A}_{d_n}, \end{array}$$

with ξ, η on coordinate s. The normal subgroup N contains the commutator $[f, x] = fxf^{-1}x^{-1} = (1, \dots, 1, [f_v, x_v], 1, \dots, 1)_n$, where:

$$\begin{aligned} [f_v, x_v] &= (1, \dots, \xi, \dots, 1)(x_{v1}, \dots, x_{vd_n})\sigma_v(1, \dots, \xi^{-1}, \dots, 1)\sigma_v^{-1}(x_{v1}^{-1}, \dots, x_{vd_n}^{-1}) \\ &= (1, \dots, 1, \xi, 1, \dots, 1, x_{vt}\xi^{-1}x_{vt}^{-1}, 1, \dots, 1), \end{aligned}$$

with ξ in coordinate s and $x_{vt}\xi^{-1}x_{vt}^{-1}$ in coordinate t. Taking another commutator, the subgroup N contains $[g, [f, x]] = (1, \dots, 1, [g_v, [f_v, x_v]], 1, \dots, 1)_n$ with:

$$[g_v, [f_v, x_v]] = (1, \dots, 1, [\eta, \xi], 1, \dots, 1),$$

and this for ξ, η in G_{n+1} arbitrary, which can be rewritten:

$$N \ni (1, \dots, 1, [\eta, \xi], 1, \dots, 1)_{n+1} \in G_{n+1} \wr Aut^e(T_{d_0\dots d_n}),$$

with $[\eta, \xi]$ in position vs. As this group is perfect, the subgroup N contains $1 \times \cdots \times G_{n+1}^{(vs)} \times \cdots \times 1$. The transitivity of the action of $Aut^e(T_{d_0\dots d_n})$ on level $n+1$ by conjugation ensures that N contains $(G_{n+1} \times \cdots \times G_{n+1}) = K_{n+1}$ as required. □

8.2. **Non subexponential amenability.** Denote SG_0 (respectively EG_0) the class of groups such that all finitely generated subgroups have subexponential growth (respectively are abelian). Assume that for an ordinal $\alpha > 0$ the classes SG_β and EG_β are defined for every ordinal $\beta < \alpha$. When α is a limit ordinal, set $SG_\alpha = \cup_{\beta<\alpha} SG_\beta$ (respectively $EG_\alpha = \cup_{\beta<\alpha} EG_\beta$). When α is a successor ordinal, define SG_α (respectively EG_α) to be the class of groups that can be obtained from groups in the class $SG_{\alpha-1}$ (respectively $EG_{\alpha-1}$) either by taking direct limits, or by taking extension by a group from the class SG_0 (respectively EG_0).

Each class SG_α (respectively EG_α) is closed under taking quotients and subgroups. Moreover, the class $SG = \cup_\alpha SG_\alpha$ (respectively $EG = \cup_\alpha EG_\alpha$) where the union runs over all ordinals α, is the smallest class of groups containing SG_0 (respectively EG_0) which is closed under the operations of taking subgroups, quotients, extensions and direct limits. As these operations preserve amenability, which is satisfied in SG_0 (respectively EG_0), the class SG (respectively EG) is called class of subexponentially (respectively elementary) amenable groups.

This construction of classes of groups is detailled in [Osi1]. It is obvious that EG_α is a subclass of SG_α for each ordinal α and that the class SG contains the class EG. This inclusion is strict (see [Gri1]) and the Basilica group introduced in [GZ] was the first example of an amenable group out of SG. Osin has shown in [Osi2] that the class EG contains no group of non uniform growth. In particular, groups in the class χ such as the groups $G(\mathcal{A}_{d_0}, A_{\bar{d}})$ of non uniform exponential growth introduced in section 7.2 are not in EG. The following Proposition shows these groups are not even in SG, providing uncountably many pairwise non isomorphic examples of amenable groups outside SG.

Proposition 8.5. *Consider a residually finite group G belonging to a class χ_k (see section 7), then one of the two following holds:*

1) *either G belongs to the class SG_0 of groups of subexponential growth,*
2) *or G does not belong to the class SG of subexponentially amenable groups.*

In particular, residually finite groups of exponential growth in a class χ_k are not in SG.

Recall an elementary property of ordinals:

Fact 8.6 (Theorem 7.3 (5) in [Kun])**.** *Let C be a non empty set of ordinals, then there exists $x \in C$ such that for every $y \in C$, one has $x \leq y$. In other words, C has a minimum.*

Proof of Proposition 8.5. The proof is similar to that in [GZ]. Let G a group in a class χ_k having exponential growth, in particular not in the class SG_0. Denote G_i the group in the class χ_k such that $G = G_0 \simeq G_i \wr \mathcal{A}_{d_{i-1}} \wr \cdots \wr \mathcal{A}_{d_0}$. In particular all groups G_i have exponential growth. Assume G_0 lies in the class SG, then all the groups G_i (which are subgroups of G_0) lie in SG. For each integer i define α_i to be the minimal ordinal for which G_i belongs to SG_{α_i} (exists by Fact 8.6). The family $\{\alpha_i\}_{i\in\mathbb{N}}$ admits a minimum α_{i_0}. Now the ordinal α_{i_0} is not a limit ordinal otherwise G_{i_0} would belong to SG_β for some $\beta < \alpha_{i_0}$. Moreover, the group G_{i_0} is not a direct limit of a strictly increasing infinite sequence of groups because it is finitely generated. This forces the existence of N and H in $SG_{\alpha_{i_0}-1}$ such that the sequence $1 \to N \to G_{i_0} \to H \to 1$ is exact. But as G hence G_{i_0} is residually finite, Proposition 8.1 implies that $N = G_{i_0+m}$ for some integer m, so that $\alpha_{i_0+m} \leq \alpha_{i_0} - 1$ which contradicts minimality of α_{i_0}, proving G is not in SG. $\square$

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